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# Un théorème limite sur la dérivée de l'intégrale de Poisson-Weierstrass généralisée

par

H. MILICER-GRUŻEWSKA

Présenté par T. WAŻEWSKI le 9 Novembre, 1957

Soit l'équation aux dérivées partielles du type parabolique

$$(1) \quad \hat{p}(u) = \sum_{\alpha, \beta=1}^n a_{\alpha\beta}(x_1, \dots, x_n, t) \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \\ + \sum_{\alpha=1}^n b_\alpha(x_1, \dots, x_n, t) \frac{\partial u}{\partial x_\alpha} + c(x_1, \dots, x_n, t) \cdot u - \frac{\partial u}{\partial t} = 0,$$

où les coefficients  $a_{\alpha\beta}$ ,  $b_\alpha$ ,  $c$  sont des fonctions des variables  $(x_1, \dots, x_n, t)$  déterminées pour  $(x_1, \dots, x_n)$  appartenant à un domaine  $\Omega$ , borné et mesurable dans l'espace euclidien à  $n$  dimensions et pour  $t$  appartenant à l'intervalle  $(0 \leq t \leq T)$ .

Supposons que la forme quadratique

$$(2) \quad \sum_{\alpha, \beta=1}^n a_{\alpha\beta} x_\alpha x_\beta$$

est définie positive dans le domaine fermé  $\Omega$  pour  $0 \leq t \leq T$ .

On suppose que les coefficients de l'équation (1) vérifient dans ce domaine les conditions d'Hölder par rapport aux variables spatiales  $A(x_1, \dots, x_n)$  et aussi par rapport à la variable  $t$  de la forme

$$(3) \quad |a_{\alpha\beta}(A, t) - a_{\alpha\beta}(A_1, t_1)| \leq k[r_{AA_1}^h + (t - t_1)^{h'}], \quad 0 < h, h' \leq 1 \\ |b_\alpha(A, t) - b_\alpha(A_1, t)| \leq k' r_{AA_1}^h, \quad \alpha, \beta = 1, \dots, n, \quad |c(A, t) - c(A_1, t)| \leq k'' r_{AA_1}^h,$$

où  $r_{AA_1}$  désigne la distance des points  $A$  et  $A_1$ ;  $k, k', k''$  sont des constantes positives.

En admettant les suppositions (2) et (3) W. Pogorzelski [1] a démontré que la solution fondamentale de l'équation (1) est une fonction donnée par la formule

$$(4) \quad \Gamma(A, t, B, \tau) = \\ = w^{B, \tau}(A, t, B, \tau) + \int_{\tau}^t \int_{\Omega'(M)} \int w^{M, \theta}(A, t, M, \theta) \Phi(M, \theta, B, \tau) dM d\theta \\ A \neq B; \quad A, B \in \Omega'; \quad 0 \leq \tau < t \leq T,$$

où

$$(5) \quad w^{M, \theta}(A, t, B, \tau) = (t - \tau)^{-\frac{n}{2}} \exp \left[ -\frac{\vartheta^{M, \theta}(A, B)}{4(t - \tau)} \right],$$

$$(6) \quad \vartheta^{M, \tau}(A, B) = \sum_{\alpha, \beta=1}^n a^{\alpha, \beta}(M, \tau) (x_{\alpha} - \xi_{\alpha})(x_{\beta} - \xi_{\beta}),$$

$a^{\alpha\beta}(M, \tau)$  sont les éléments de la matrice inverse à la matrice  $\|a_{\alpha\beta}(M, \tau)\|$ ,  $A(x_1, \dots, x_n)$  et  $B(\xi_1, \dots, \xi_n)$  sont les deux points arbitraires du domaine  $\Omega$ ;  $\Omega'$  est un domaine arbitraire, borné et mesurable, qui contient  $\Omega$  avec son bord  $S$ . On a prolongé tous les coefficients en dehors du domaine  $\Omega$  au domaine  $\Omega'$  de façon que les suppositions (2) et (3) soient vérifiées.

La condition (2) implique l'existence des deux nombres positifs  $g$  et  $G$  tels que:

$$(5) \quad 4gr_{AB}^2 < \vartheta^{M, \tau}(A, B) < 4Gr_{AB}^2.$$

La fonction  $\Phi(A, t, B, \tau)$  est la solution d'une équation intégrale de Volterra faiblement singulière:

$$(7) \quad \Phi(A, t, B, \tau) = f(A, t, B, \tau) + \lambda \int_{\tau}^t \int_{\Omega'(M)} \int N(A, t, M, \theta) \Phi(M, \theta, B, \tau) dM d\theta,$$

où

$$(8) \quad N(A, t, M, \theta) = \sqrt{\det |a^{\alpha\beta}(A, t)|} \hat{\varphi}_{A, t}[w^{M, \theta}(A, t, M, \theta)],$$

$$(8') \quad f(A, t, B, \tau) = \lambda N(A, t, B, \tau), \quad \lambda = (2/\sqrt{\pi})^{-n},$$

$$(8'') \quad \Phi(A, t, B, \tau) = \\ = f(A, t, B, \tau) + \lambda \int_{\tau}^t \int_{\Omega'(M)} \int \Re(A, t, M, \theta) f(M, \theta, B, \tau) dM d\theta,$$

où

$$(8''') \quad \Re(A, t, M, \theta) = N(A, t, M, \theta) + \sum_{v=1}^n \lambda^v N_v(A, t, M, \theta)$$

est le noyau résolvant de l'équation (8). Les éléments de la série (8) sont des noyaux itérés du noyau  $N(A, t, M, \theta)$ .

## L'intégrale de volume

$$(9) \quad I(A, t, \tau) = \int \int_{\Omega} \Gamma(A, t, B, \tau) \varrho(B, \tau) dB,$$

sera dite l'intégrale de Poisson-Weierstrass généralisée;  $\varrho(B, \tau)$  est une fonction bornée et continue pour  $B \subset \Omega$ ,  $0 < \tau \leq T$ .

W. Pogorzelski a prouvé dans son travail [2] que

$$(10) \quad \lim_{\tau \rightarrow t} \mathcal{J}(A, t, \tau) = (2 \sqrt{\pi})^n \varrho(A, t) / \sqrt{\det |a^{\alpha\beta}(A, t)|}, \quad (A \subset \Omega),$$

sous les suppositions (2), (3) et (9).

J'ai démontré le théorème limite suivant sur la dérivée de l'intégrale (9).

$$(11) \quad \lim_{\tau \rightarrow t} \mathcal{J}_{x_a}(A, t, \tau) = \left[ (2 \sqrt{\pi})^n \varrho(A, t) / \sqrt{\det |a^{\alpha\beta} a(A, t)|} \right]'_{x_a},$$

( $A \in \Omega$ ) que m'a posé W. Pogorzelski. J'ai supposé que les fonctions  $a_{\alpha\beta}(B, \tau)$  et  $\varrho(B, \tau)$  admettent les premières dérivées spatiales continues dans le point  $(A, t)$ . Cette démonstration sera publiée séparément.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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# A Generalization of Two Norm Spaces. Linear Functionals

by

A. ALEXIEWICZ and Z. SEMADENI

*Presented by W. ORLICZ on December 16, 1957*

1. Let  $\langle X, \|\cdot\|, \tau \rangle$  be a triple in which  $X$  is a linear space,  $\|\cdot\|$  — a (homogeneous) pseudonorm in  $X$ , and  $\tau$  — a locally convex linear topology in  $X$ , coarser than the topology induced by the norm  $\|\cdot\|$ ; moreover, let the ball  $S = \{x: \|x\| \leq 1\}$  be closed in the space  $\langle X, \tau \rangle$ . We shall call  $\langle X, \|\cdot\|, \tau \rangle$  the space with a mixed topology. Such generalizations of two norm spaces [1] were considered first by A. Wiweger [7].

The pseudonorms continuous with respect to the topology  $\tau$  will be denoted by  $[x]_a$ , and  $\{[\cdot]_a\}_{a \in A}$  will denote the totality of all these pseudonorms. A distributive functional  $\xi$  on  $X$  will be called

$\gamma_s$ -linear, if  $\lim_{n \rightarrow \infty} [x_n]_a = 0$  for all  $a \in A$  together with  $\sup_{n=1,2,\dots} \|x_n\| < \infty$

implies  $\xi(x_n) \rightarrow 0$ ,

$\gamma_t$ -linear, if it is continuous on  $S$  with respect to the topology  $\tau$ .

$\Xi_{\gamma_s}$  and  $\Xi_{\gamma_t}$  will denote the sets of  $\gamma_s$ - and  $\gamma_t$ -linear functionals respectively;  $\Xi$  and  $\Xi_\tau$  will stand for the class of functionals continuous with respect to the norm  $\|\cdot\|$  and the topology  $\tau$ , respectively. Obviously

$$\Xi_\tau \subset \Xi_{\gamma_t} \subset \Xi_{\gamma_s} \subset \Xi.$$

We have, in general,  $\Xi_{\gamma_t} \neq \Xi_{\gamma_s}$  as the following example shows. Let  $X_0$  be the space of all bounded measurable functions  $x = x(t)$  in  $[0, 1]$ ,

$$(1) \quad \|x\| = \sup_{0 \leq t \leq 1} |x(t)|, \quad [x]_t = |x(t)| \quad (t \in [0, 1]).$$

Each  $\gamma_t$ -linear functional on  $X_0$  is of the form

$$(2) \quad \xi(x) = \sum_{v=1}^{\infty} a_v x(t_v)$$

with fixed  $t_1, t_2, \dots$  and  $\sum_{v=1}^{\infty} |a_v| < \infty$ . On the other hand, the functional

$\int_0^1 x(t) dt$  is obviously  $\gamma_s$ -linear and is not in  $\Xi_{\gamma_t}$ .

Under Ulam's hypothesis (that there are no inaccessible alephs between  $\aleph_0$  and  $2^{\aleph_0}$ ) [6] it may be shown that each  $\gamma_s$ -linear functional on the above space is of the form

$$\xi(x) = \xi_0(x) + \int_0^1 x(t) \varphi(t) dt,$$

where  $\xi_0 \in \mathcal{E}_{\gamma_t}$  and  $\int_0^1 |\varphi(t)| dt < \infty$ . However, if  $X$  is the space of all bounded functions in  $[0, 1]$  with the same  $\|\cdot\|$  and  $[\cdot]_t$ , then, also under Ulam's hypothesis, the spaces  $\mathcal{E}_{\gamma_s}$  and  $\mathcal{E}_{\gamma_t}$  coincide and consist of all the functionals of form (2). A comparison of this space with the preceding one as its subspace (sequentially closed in the topology  $\tau$ ) enables us to conclude that the  $\gamma_s$ -linear functionals do not have the extension property.

By  $\|\xi\|$  we shall denote the norm of  $\xi$  in  $\mathcal{E}$ , i. e.

$$\|\xi\| = \sup \{ \xi(x) : \|x\| \leq 1 \}.$$

**THEOREM 1.** *The class  $\mathcal{E}_{\gamma_t}$  is identical with the closure of  $\mathcal{E}_{\tau}$  in the space  $\langle \mathcal{E}, \|\cdot\| \rangle$ , equivalently, each  $\gamma_t$ -linear functional is of the form*

$$(3) \quad \xi(x) = \sum_{n=1}^{\infty} \zeta_n(x),$$

where  $\zeta_n \in \mathcal{E}_{\tau}$  and  $\sum_{n=1}^{\infty} \|\zeta_n\| < \infty$ .

The proof of this theorem does not differ from that of Theorems 4.2 and 4.3 of [2].

The following theorem makes it possible to restrict considerations of denumerable sets of  $\gamma_t$ -linear functionals to the case of normable topology  $\tau$ .

**THEOREM 2.** *For each sequence  $\xi_1, \xi_2, \dots$  of  $\gamma_t$ -linear functionals there exists a sequence  $[\cdot]_{a_n}$  of pseudonorms and a sequence  $b_n$  of positive numbers such that*

$$\|x\|^* = \sum_{k=1}^{\infty} b_k [x]_{a_k} < \infty \quad \text{for all } x \in X,$$

and such that

$$\sup_{n=1,2,\dots} \|x_n\| < \infty, \quad \|x_n\|^* \rightarrow 0 \quad \text{imply} \quad \xi_m(x_n) \rightarrow 0 \quad \text{for } m = 1, 2, \dots$$

**Proof.** Let us represent  $\xi_m$  as  $\xi_m = \sum_{k=1}^{\infty} \zeta_{km}$ , where  $\zeta_{km} \in \mathcal{E}_{\tau}$  and

$\sum_{k=1}^{\infty} \|\zeta_{km}\| < \infty$  ( $m = 1, 2, \dots$ ). Then  $|\zeta_{km}(x)| \leq [x]_{\beta_{km}}$  with certain  $\beta_{km} \in A$ .

By arranging the pseudonorms  $[ \cdot ]_{k_m}$  into a single sequence  $[ \cdot ]_{k_k}$  we have  $[x]_{a_k} \leq M_k \|x\|$  (because  $\tau$  is coarser than  $[ \cdot ]_{k_k}$ ) and, setting  $b_k = (2^k M_k)^{-1}$ , we are led to the conclusion of the theorem.

2. Some applications of Theorem 1 in the case of metrisable topology  $\tau$  are given in [2]. We shall now present applications in the case of non-metrisable  $\tau$ .

A. Let  $T$  be a completely regular Hausdorff space, let  $X$  be the space  $C^*(T)$  of bounded, real-valued, continuous functions  $x = x(t)$  on  $T$  with the norm  $\|x\| = \sup \{|x(t)| : t \in T\}$ . Let  $\{T_\beta\}_{\beta \in B}$  be a family of (not necessarily all) compact subsets of  $T$ , and let  $\tau$  be the topology determined by the pseudonorms  $[x]_\beta = \sup \{|x(t)| : t \in T_\beta\}$ . By Theorems 2 and 1, for each  $\gamma_l$ -linear functional  $\xi$  on  $\langle X, [ \cdot ]_\beta, \tau \rangle$  there exists a sequence  $\beta_n \in B$  such that

$$x(t) = y(t) \quad \text{on} \quad \Theta = \bigcup_{n=1}^{\infty} T_{\beta_n} \quad \text{implies} \quad \xi(x) = \xi(y)$$

and  $\xi$  is of form

$$\xi(x) = \int_{\Theta} x(t) d\mu,$$

$\mu$  being a signed Borel measure. This theorem was proved by J. Mařík ([4], p. 90, see also [2]).

B. Let  $X$  be the space conjugate to a Banach space  $Z$ ;  $[ \cdot ]$  will denote the usual norm of elements of  $X$  as linear functionals and  $\tau$  will be the weak topology  $\sigma(X, Z)$ . In this case  $\Xi$  is the second conjugate to  $Z$ ,  $\Xi_\tau$  is the canonical image of  $Z$ , which is closed in  $\langle \Xi, [ \cdot ], \tau \rangle$ . Hence, the general form of  $\gamma_l$ -linear functionals in  $\langle X, [ \cdot ], \tau \rangle$  is  $\xi(x) = x(z)$  with  $z$  independent of  $x$ , and  $\Xi_\tau = \Xi_{\gamma_l}$ .

3. Now, let  $X$  be both a space with mixed topology and a vector lattice. Let us suppose that

$$\begin{aligned} x \wedge y = 0 & \quad \text{implies} \quad \|x + y\| = \|x - y\|, \quad [x + y]_a = [x - y]_a \quad (a \in A), \\ 0 \leq x \leq y & \quad \text{implies} \quad \|x\| \leq \|y\|, \quad [x]_a \leq [y]_a \quad (a \in A). \end{aligned}$$

All the spaces  $\Xi_\tau, \Xi_{\gamma_l}, \Xi_{\gamma_s}, \Xi$  are vector lattices with the natural ordering; the operation  $\vee$  is then defined by the well-known formula of F. Riesz [5],

$$(\xi \vee \eta)(x) = \sup \{\xi(x_1) + \eta(x_2) : x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + x_2 = x\},$$

when  $x \geq 0$ , and, in the case of arbitrary  $x$ ,

$$(\xi \vee \eta)(x) = (\xi \vee \eta)(x_+) - (\xi \vee \eta)(x_-) \quad \text{where} \quad x_- = (-x)_+ = (-x) \vee 0.$$

A subspace  $X_0$  of  $X$  is called contracted or  $l$ -ideal (Birkhoff [3], p. 222) if  $X_0$  is linear and  $x \in X_0$ ,  $|y| \leq |x|$  implies  $y \in X_0$ .

LEMMA 1. Let  $X_0$  be a contracted subspace of  $X$  and let  $\xi, \eta \in \Xi$  be such that  $\xi(x) = \eta(x)$  for  $x \in X_0$ . Then  $(\xi \vee \eta)(x) = (\xi \wedge \eta)(x) = \xi(x)$  for  $x \in X_0$ .

LEMMA 2. Let  $X_0$  be as above, and let  $\zeta$  be a non-negative linear functional on  $\langle X_0, [\ ]_a \rangle$ ,  $a$  being fixed. There then exists an extension  $\bar{\zeta} \in \Xi_\tau$  of  $\zeta$  satisfying the conditions

$$\|\bar{\zeta}\|_{X_0} = \|\zeta\|_{X_0},$$

$$\sup \{\bar{\zeta}(x) : [x]_a \leq 1, x \in X_0\} = \sup \{\bar{\zeta}(x) : [x]_a \leq 1, x \in X\}.$$

Proof. Let  $\zeta_1$  be the Hahn-Banach extension of  $\zeta$  from  $\langle X_0, \|\ \| \rangle$  onto  $\langle X, \|\ \| \rangle$  and let analogously  $\zeta_2$  be that of  $\zeta$  from  $\langle X_0, [\ ]_a \rangle$  onto  $\langle X, [\ ]_a \rangle$ . Setting

$$\zeta = (\zeta_1 \vee 0) \wedge (\zeta_2 \vee 0),$$

we obtain a functional  $\bar{\zeta}$  defined on  $X$ , which is an extension of  $\zeta$  (by Lemma 1). Moreover,  $\|\bar{\zeta}\| \leq \|\zeta_1 \vee 0\| \leq \|\zeta_1\| = \|\zeta\|$  and

$$\begin{aligned} \sup \{\bar{\zeta}(x) : [x]_a \leq 1\} &\leq \sup \{(\zeta_2 \vee 0)(x) : [x]_a \leq 1\} \leq \sup \{\zeta_2(x) : [x]_a \leq 1\} = \\ &= \sup \{\zeta(x) : [x]_a \leq 1, x \in X_0\}. \end{aligned}$$

THEOREM 3. Let  $\xi$  be a  $\gamma_l$ -linear functional on a contracted subspace  $X_0$  of  $X$ . There then exists a  $\gamma_l$ -linear extension  $\bar{\xi}$  of  $\xi$  on  $X$ .

Proof. Let us represent  $\xi$  in the form (3);  $\zeta_n \in \Xi_\tau$  implies  $|\zeta_n^-(x)| \leq [x]_{a_n}$ ,  $|\zeta_n^+(x)| \leq [x]_{a_n}$  for some  $a_n \in A$ . We extend  $\zeta_n^+$  and  $\zeta_n^-$  according to Lemma 2 to functionals  $\eta'_n$  and  $\eta''_n$  respectively.

Then

$$\|\eta'_n\| = \|\zeta_n^+\| \leq \|\zeta_n\|, \quad \text{and} \quad \|\eta''_n\| = \|\zeta_n^-\| \leq \|\zeta_n\|,$$

whence

$$\sum_{n=1}^{\infty} \|\eta'_n + \eta''_n\| < \infty.$$

The functionals  $\eta'_n, \eta''_n$  ( $n = 1, 2, \dots$ ) being in  $\Xi_\tau$ , the functional  $\bar{\xi} = \sum_{n=1}^{\infty} (\eta'_n + \eta''_n)$  is the desired extension.

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# Note on Some Function-Theoretic Inequalities

by  
OU ŠO-MO

*Presented by H. STEINHAUS on December 30, 1957*

In this note certain results are enounced concerning inequalities satisfied by analytic functions; the details of proofs will be published elsewhere.

**THEOREM 1.** *Suppose  $D$  is a bounded region of finite connectivity. Assume the boundary  $\Gamma$  of  $D$  piecewise smooth and the inner angles of  $\Gamma$  at its corner points larger than zero, and let  $f(\zeta)$  be a function analytic for  $\zeta \in D$ . If  $\operatorname{Im} f(\zeta_0) = 0$ ,  $\zeta_0 \in D$ ,  $p \geq 1$ , then no matter how small the positive constant  $\varepsilon$ , we have*

$$\left\{ \iint_D |f(\zeta)|^p dx dy \right\}^{1/p} \leq C_1 \left\{ \iint_D \operatorname{Re} |f(\zeta)|^{p+\varepsilon} dx dy \right\}^{1/(p+\varepsilon)} \quad (\zeta = x + iy);$$

here the positive constant  $C_1 = C_1(D, \zeta_0; p, \varepsilon)$  is independent of  $f$ .

This theorem is sharper than that of Ivo Babuška [1], but our proof is more elementary and simpler.

For a smooth arc  $L: \zeta = \zeta(s)$ ,  $0 \leq s \leq 1$ ,  $s$  — the natural parameter of  $L$ , let us denote by  $\theta(s)$ ,  $0 \leq s \leq 1$ , the angle formed by the tangent to  $L$  at the point  $\zeta(s)$  together with the real axis. If the modulus of continuity  $\omega(t)$ ,  $t > 0$ , of  $\theta(s)$  satisfies the condition:  $\int_0^\delta \frac{\omega(t)}{t} dt < \infty$ ,  $\delta$  — some positive const., then we say that  $L$  belongs to class (A). If  $\theta^{(n)}(s)$ ,  $n = 0, 1, 2, \dots$ , is continuous on  $0 \leq s \leq 1$ , then arc  $L$  is said to belong to class  $(S^{(n)})$ .

**THEOREM 2.** *Let  $D$  be a bounded region of finite connectivity, with piecewise smooth boundary  $\Gamma$ , and the inner angles of  $\Gamma$  at its corner points larger than zero. If  $f(\zeta)$  is an analytic function, defined in  $D$ , such that*

$$f(\zeta) \neq 0, \quad |\arg f(\zeta)| < \frac{\pi}{2\lambda}, \quad 0 < \lambda < \infty, \quad \zeta \in D,$$

then

$$f(\zeta) \in H_p(D), \quad 0 < p < \lambda,$$

(i. e.  $\iint_D |f(\zeta)|^p dx dy < \infty$ ), and

$$\left\{ \iint_D |f(\zeta)|^p dx dy \right\}^{1/p} \leq C_2 \{ \operatorname{Re}[f^{p+\varepsilon}(\zeta_0)] \}^{1/p+\varepsilon};$$

here,  $\zeta_0 \in D$ ,  $\varepsilon > 0$ , and  $\operatorname{const.} C_2 = C_2(D, \zeta_0, p, \varepsilon, \lambda)$  is independent of  $f(\zeta)$ . Furthermore, if every smooth arc of  $\Gamma$  belongs to class (A), and the inner angles of  $\Gamma$  at its corner points are larger than  $\pi$ , then, in the above inequality, the value 0 for  $\varepsilon$  is not excluded.

**THEOREM 3.** If  $D$  is a simply connected region, bounded by a piecewise smooth Jordan curve  $\Gamma$ , if the inner angles of  $\Gamma$  at its corner points are larger than zero, and the function  $f(\zeta)$  satisfies the conditions of Theorem 2, then  $f(\zeta) \in E_p(D)$  for every  $0 < p < \lambda$ , (namely,  $\int_{\Gamma_r} |f(\zeta)|^p d\zeta < M_p < \infty$ ;

here  $\Gamma_r$ ,  $0 < r < l$ , are the inner level curves of  $D$  and the  $\operatorname{const.} M_p$  is independent of  $r$ ; for some properties concerning the class of functions  $E_p(D)$ , see [2]), and

$$\left\{ \int_{\Gamma} |f(\tilde{\zeta})|^p ds \right\}^{1/p} \leq C_3 \{ \operatorname{Re}[f^{p+\varepsilon}(\zeta_0)] \}^{1/p+\varepsilon} \quad \tilde{\zeta} \in \Gamma, \quad \varepsilon > 0,$$

for any fixed  $\zeta_0 \in D$ ; here  $s$  is the natural parameter of  $\Gamma$ ,  $\operatorname{const.} C_3 = C_3(D, \zeta_0, p, \varepsilon)$  is independent of  $f(\zeta)$ , and  $f(\tilde{\zeta})$  is defined on  $\Gamma$  by the angular limit value of  $f(\zeta)$ . Moreover, if every smooth arc of  $\Gamma$  belongs to class (A) and the inner angles of  $\Gamma$  at its corner points are larger than  $\pi$ , then we can give to the  $\operatorname{const.} \varepsilon$  in the above inequality also the value 0.

**THEOREM 4.** If the boundary  $\Gamma$  of a bounded simply connected region  $D$  is a smooth Jordan curve, then, for an analytic function  $f(\zeta)$  defined in  $D$ ,  $f(\zeta) \in E_p(D)$  ( $p > 0$ ) implies  $f(\zeta) \in H_{p_1}(D)$  ( $0 < p_1 < p$ ), and

$$\left\{ \iint_D |f(\zeta)|^{p_1} dx dy \right\}^{1/p_1} \leq C_4 \left\{ \int_{\Gamma} |f(\tilde{\zeta})|^p ds \right\}^{1/p}, \quad \tilde{\zeta} \in \Gamma;$$

here  $\operatorname{const.} C_4$  is independent of  $f(\zeta)$ ,  $s$  is the natural parameter of  $\Gamma$ , and  $f(\tilde{\zeta})$  is the angular limit value of  $f(\zeta)$  on  $\Gamma$ . Furthermore, if  $\Gamma \in (A)$ , then, in the above inequality, we can put  $p_1 = p$ .

Concerning an analytic function and its derivatives, we obtain

**THEOREM 5.** Let  $D$  be a bounded region of finite connectivity; if its boundary  $\Gamma$  is piecewise smooth, if every smooth arc belongs to class  $s^{(n-1)}$ ,  $n = 1, 2, \dots$ , and the inner angles of  $\Gamma$  at its corner points are larger than  $\pi$ , then

$$\left\{ \iint_D |f^{(n)}(\zeta)|^a dx dy \right\}^{1/a} \leq C_5 \left\{ \iint_D |\operatorname{Re} f(\zeta)|^p dx dy \right\}^{1/p};$$

here,  $p > 1$ ,  $0 < \alpha < \frac{p}{np+2}$ , and the positive const.  $C_5 = C_5(p, n, \alpha, D)$  is independent of  $f(\zeta)$ . Moreover, if every smooth arc of  $\Gamma$  belongs to class (A), then we can put  $p = 1$ ; when  $\Gamma$  is only piecewise smooth, (i. e. the case  $n = 1$ ), then the constant  $\pi$  mentioned in condition "the inner angles of  $\Gamma$  at its corner points, are larger than  $\pi$ ", can be replaced by 0.

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# A Uniqueness Condition with a Standard Differential Equation without Uniqueness Property

by

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In this note we shall give a new sufficient condition for the (one-sided) uniqueness of the solutions of the initial problem for a system of ordinary differential equations (written in the vector form)

$$(1) \quad X' = F(t, X),$$

where

$$X = (x_1, \dots, x_n), \quad F = (f_1, \dots, f_n).$$

The majority of known uniqueness conditions for (1) are based (explicitly or not) on comparison with a standard differential equation of the form

$$(2) \quad x' = g(t, x),$$

which possesses the property of uniqueness.

In our uniqueness condition (Theorem 3) the standard differential equation may not possess the uniqueness property. The uniqueness property relative to  $g$  will be replaced by a more general condition (Property P). In the case of two independent variables we give a more general result (Theorem 2).

**Definition 1.** An integral  $X = X(t)$ , ( $a < t < \beta$ ), of system (1) is termed unique to the right if, for any integral  $Y = Y(t)$  of (1) defined in an interval  $\gamma \leq t \leq \delta$ ,  $a < \gamma < \delta < \beta$  and satisfying the equality

$$Y(\gamma) = X(\gamma),$$

we have

$$Y(t) = X(t) \quad \text{for} \quad \gamma \leq t \leq \delta.$$

**PROPERTY P.** We say that a function  $g(t, x)$  defined in a subset  $S$  of half-plane  $x > 0$  has Property **P**, if, for every real number  $s$ , there exists a sequence  $x_n(t)$  of positive solutions of (2) such that:

I. either there exists a positive number  $\varepsilon$  such that

$$\lim_{n \rightarrow \infty} x_n(t) = 0 \quad \text{for} \quad s \leq t \leq s + \varepsilon,$$

II. or there exists a sequence of numbers  $t_n$  such that  $t_n > s$  ( $n = 1, 2, \dots$ ),  $t_n \rightarrow s$  as  $n \rightarrow \infty$  and, for each fixed  $n$ ,

$$x_n(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow t_n \quad (s < t < t_n).$$

**Remark.** It is evident that, if the function  $g(t, x)$  is continuous in the set  $0 \leq x \leq \eta(t)$ , where  $\eta(t)$  is a positive continuous function defined for  $-\infty < t < +\infty$ ,  $g(t, 0) \equiv 0$ , and the solution  $x = 0$  of (2) is unique to the right, then the function  $g(t, x)$  considered in the set  $S$ :  $0 < x < \eta(t)$  has Property **P**. The following example shows that  $g(t, x)$  may possess Property **P** without having the property of right-sided uniqueness.

**EXAMPLE 1.** The right-hand member of the differential equation

$$(3) \quad x' = 3(t^2 - |x - t^3|^{2/3})$$

possesses Property **P**, but its solution  $x = 0$  is not unique to the right. Indeed, the function  $x = t^3$  is also a solution of this equation and vanishes for  $t = 0$ , hence the solution  $x = 0$  is not unique to the right.

To prove Property **P**, we take  $x_n(t) = t^3 + \left(\frac{1}{n} - t\right)^3$ . The differential equation with analogous properties, for which  $g(t, x) > 0$  for  $x > 0$ , may be constructed.

**LEMMA.** Suppose that  $g(t, x)$  has Property **P** and that the function  $v(t)$  has the following properties:

$$\begin{aligned} v(t) &\text{ is continuous for } s \leq t \leq r \\ v(s) &\leq 0 \end{aligned}$$

and

$$(4) \quad D_- v(t) < g(t, v(t))$$

if  $s < t < r$ ,  $(t, v(t)) \in S$ , where  $D_-$  denotes the left lower derivative.

Then  $v(t) \leq 0$  for  $s \leq t \leq r$ .

**Proof.** Suppose the contrary. Hence, there exists such an interval  $\langle p, q \rangle$  that  $s \leq p < q \leq r$ ,  $v(p) = 0$  and  $v(t) > 0$  for  $p < t \leq q$ . It results from Property **P** that there exists a solution  $x(t)$  of (2) and such a number  $c$  that  $p < c < q$ ,  $x(p) > v(p)$ , and  $x(c) < v(c)$ , the solution  $x(t)$  being positive in the interval  $p \leq t \leq c$ . Hence  $x(t) = v(t)$  for some  $t$ . Let us

denote by  $b$  the smallest number  $t$ , such that  $x(t) = v(t)$ ,  $p < t < c$ . We have

$$\frac{v(t) - v(b)}{t - b} > \frac{x(t) - x(b)}{t - b}$$

for  $p < t < b$ . It follows that

$$x'(b) \leq D_- v(b) < g(b, v(b)) = g(b, x(b))$$

and, by (2),  $x'(b) = g(b, x(b))$ . This contradiction completes the proof.

**Definition 2.** An integral  $x = x(t)$ ,  $\alpha < t < \beta$ , of the differential equation

$$(5) \quad x' = f(t, x),$$

where  $f(t, x)$  is defined in a plane set  $D$ , is termed maximal integral to the right if, for any integral of (5)  $y = y(t)$  defined in an interval  $\gamma \leq t \leq \delta$ ,  $\alpha < \gamma < \delta < \beta$ , and satisfying the equality

$$y(\gamma) = x(\gamma),$$

we have

$$y(t) \leq x(t)$$

for  $\gamma \leq t \leq \delta$ .

**THEOREM 1.** The sufficient condition for a given integral  $H: x = h(t)$  of (5), where  $f(x, y)$  is defined in an open set  $D$  ( $H \in D$ ), to be a maximal integral to the right, is the existence of a function  $g(t, x)$  possessing Property **P** and satisfying the inequality

$$(6) \quad f(t, y) - f(t, x) < g(t, y - x)$$

for  $(t, x) \in H$ ,  $(t, y) \in D$ ,  $(t, y - x) \in S$ .

The proof is based on the Lemma. We take  $v(t) = y(t) - h(t)$ . The differential inequality (4) results from (6).

**Remark 2.** It follows from the known comparison theorems for differential equation (5) that the existence of such a function  $g(t, x)$  is also necessary for the integral  $H$  to be a maximal integral to the right of (5), if the function  $f(t, x)$  is continuous.

**EXAMPLE 2.** It results from Theorem 1 and Example 1 that, for  $f(x, y) = 3(t^2 - |x - t^{2/3}|) - k(x)$ ,

where  $k(x)$  ( $-\infty < x < +\infty$ ) is an arbitrary continuous function,  $k(0) = 0$  and  $k(x) > 0$  for  $x > 0$ , the integral  $x = 0$  of (5) is a maximal integral to the right. We take the right-hand member of (3) for  $g(t, x)$ .

The following theorem is a corollary of Theorem 1.

**THEOREM 2.** Suppose that the function  $f(t, x)$  defined in an open plane set  $D$  satisfies inequality (6) for  $(t, x) \in D$ ,  $(t, y) \in D$ ,  $(t, y - x) \in S$ , where the function  $g(t, x)$  fulfills Property **P**.

Then, every solution  $x = x(t)$  (contained in  $D$ ) of Eq. (5) is unique to the right.

**THEOREM 3.** Suppose that the right-hand members of system (1), defined in an open set  $R$ , satisfy the inequality

$$|F(t, X) - F(t, Y)| < g(t, |X - Y|)$$

for  $(t, X) \in R$ ,  $(t, Y) \in R$ ,  $(t, |X - Y|) \in S$ ,

$|A| = (a_1^2 + \dots + a_n^2)^{1/2}$ , where the function  $g(t, x)$  satisfies Property **P**.

Then, every integral of (1) is unique to the right.

Theorem 3 results from the Lemma, if we take  $r(t) = |X(t) - Y(t)|$  and note that  $D_-|X(t) - Y(t)| \leq |X'(t) - Y'(t)| = |F(t, X(t)) - F(t, Y(t))| < g(t, |X(t) - Y(t)|)$  for  $(t, |X(t) - Y(t)|) \in S$ .

Theorem 1 becomes false if we replace the strong inequality (6) by the weak inequality  $f(t, y) - f(t, x) \leq g(t, y - x)$ .

This applies in like manner to the Lemma.

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# Entwicklung positiv definiter Kerne nach Eigendistributionen. Differenzierbarkeit der Spektralfunktion eines hypoelliptischen Operators

von

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In einer vor Kurzem publizierten Note kündigte Berezanskij einen sehr interessanten Satz an [1]. Mit diesem Satz gelang es ihm viele Ergebnisse über die Darstellung positiv definiter Funktionen unter einen Hut zu bringen, so z. B. den klassischen Satz von Bochner, entsprechende Sätze von Povzner, M. G. Krein und einen Satz von S. N. Bernstein.

Es scheint uns jedoch, dass der Satz nicht allgemein genug ausgesprochen war. Deshalb zeigen wir in dieser Arbeit, wie man durch die Anwendung einer Realisation des vollständigen Spektralsatzes [8] einerseits und der Theorie der Schwartzschen Kerne [6] und des Satzes von abgeschlossenem Graphen andererseits aufs Einfachste eine weitgehende Verallgemeinerung des Satzes von Berezanskij erhalten kann.

In solcher Fassung erhält der hier bewiesene Satz als Spezialfälle die Ergebnisse von Berezanskij (stetige Kerne und elliptischer Operator) und eine Verallgemeinerung eines Satzes von Gårding [9] (Diracsche  $\delta$ ) über die Differenzierbarkeit der Spektralfunktion einer selbstadjungierten Fortsetzung eines elliptischen Operators.

Unser Satz gilt auch für Kerne auf dem Doppelnebenklassenraum einer separablen Lieschen Gruppe, was wichtige Anwendungen in der Theorie der allgemeinen sphärischen Funktionen zulässt.

## Definitionen und Hilfsmittel

Es sei  $\mathfrak{H}$  ein separabler Hilbertscher Raum mit dem Skalarprodukt  $(,)$  und es sei  $\mathcal{A}$  eine  $C^*$ -Algebra der Operatoren auf  $\mathfrak{H}$ . Dann besagt der vollständige Spektralsatz (vgl. z. B. [4]), dass  $\mathcal{A}$  eine — bis auf Äquivalenz eindeutige — Zerlegung des Raumes  $\mathfrak{H}$  in ein direktes Integral  $\int_A \mathfrak{H}_\lambda d\mu(\lambda)$  induziert ( $\int_A \mathfrak{H}_\lambda d\mu(\lambda)$  ist die Menge der  $\mu$ -messbaren, quadra-

tisch integrierbaren Vektorfelder  $\hat{u}(\cdot)$ , wo  $\hat{u}(\lambda) \in \mathfrak{H}_\lambda$  mit dem Skalarprodukt  $((\hat{u}, \hat{v})) \stackrel{\text{df}}{=} \int_A (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) d\mu(\lambda)$ , wobei  $(\cdot, \cdot; \lambda)$  das skalare Produkt im Raume  $\mathfrak{H}_\lambda$  bedeutet).

Es sei  $f \mapsto A_f$  der Gelfandsche Isomorphismus. Es gibt dann eine unitäre Abbildung  $F: \mathfrak{H} \rightarrow \int \mathfrak{H}_\lambda d\mu(\mathfrak{H} \ni u \rightarrow Fu = \hat{u} \in \int \mathfrak{H}_\lambda d\mu(\lambda))$ , die die Algebra  $\mathcal{A}$  diagonalisiert, d. h. es gilt

$$(A_f, u, v) = \int_A f(\lambda) (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) d\mu(\lambda)$$

identisch für  $u, v \in \mathfrak{H}$ .

Die Menge  $\Lambda$  ist der Raum der maximalen Ideale (Charaktere) von  $\mathcal{A}$  und heisst das Spektrum von  $\mathcal{A}$ .  $Fu(\lambda) = \hat{u}(\lambda)$  heisst die Fourier-Transformierte von  $u$ .

Wenn die Algebra  $\mathcal{A}$  von einem selbstadjungierten (oder normalen) Operator  $A$  erzeugt wird, dann kann man die Menge  $\Lambda$  mit dem Spektrum  $\text{sp} A$  des Operators  $A$  identifizieren. Man hat dann insbesondere folgende Formeln

$$(1) \quad (Au, u) = \int_{\text{sp} A} \lambda (\hat{u}(\lambda), \hat{u}(\lambda); \lambda) d\mu(\lambda),$$

$$(2) \quad (u, v) = \int_{\text{sp} A} (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) d\mu(\lambda).$$

Es sei  $S = C_0^\infty(\Omega_n)$  die Menge der beliebig oft differenzierbaren Funktionen mit kompakten Trägern auf der  $n$ -dimensionalen differenzierbaren Mannigfaltigkeit  $\Omega_n$ . Es sei  $(u, v)$  ein Skalarprodukt auf  $S$ , d. h.  $(\cdot, \cdot)$  ist eine hermitesche positiv definite bilineare Form auf  $S \times S$ , die stetig bei der Schwartzschen Topologie in  $\mathcal{D}'(\Omega_n)^*$  ist ( $m = 0, 1, \dots, \infty$ ). Nach der Schwartzschen Theorie der Kerne [6] definiert  $(u, v)$  eine einzige Distribution  $K$  auf  $\Omega_n \times \Omega_n$  der Ordnung  $m$  — den sgn. Kern von  $(\cdot, \cdot)$  — so dass

$$(u, v) = \langle Ku, v \rangle \quad (K: \mathcal{D}^m \rightarrow \mathcal{D}'^m).$$

$\langle T, u \rangle$  bedeutet den Wert der Distribution  $T$  in  $u$ ; falls  $T$  lokal integrierbar ist, dann bedeutet  $\langle T, u \rangle = \int_{\Omega_n} T(x) \overline{u(x)} dx = (T, u)_0$ .

Das kann in der Form eines „Integrals“ geschrieben werden:

$$(3) \quad (u, v) = \int_{\Omega_n \times \Omega_n} K(x, y) u(x) \overline{v(y)} dx dy, \quad K(\cdot, \cdot) \in \mathcal{D}'^m(\Omega_n \times \Omega_n).$$

\*) Die Folge  $u_\nu \xrightarrow{\mathcal{D}^m} 0$ , wenn die Träger von  $u$  in einer kompakten Menge enthalten sind und  $u_\nu \rightarrow 0$  gleichmässig mit allen Ableitungen bis zur  $m$ -ter Ordnung.

Es sei  $\mathfrak{H}$  die Vervollständigung von  $\mathcal{S}$  in der Norm  $\|u\|^2 = (u, u)$ . Ein Satz von Gårding [8] besagt, dass in diesem konkreten Fall  $\hat{u}(\lambda) = Fu(\lambda) \in \mathfrak{H}_\lambda$  eine vektorielle Distribution der Ordnung  $Q \leq n + m$  ist.

### Der Hauptsatz und seine Korolläre

Nun können wir den folgenden Satz aussprechen.

**SATZ 1.** *Es sei  $(\cdot, \cdot)$  ein Skalarprodukt auf  $C_0^\infty(\Omega_n)$  und  $K$  der zugehörige positivdefinite Kern:  $(u, v) = (Ku, v)$ . Es sei  $L: \mathcal{S} \rightarrow \mathcal{D}'$  ein in bezug auf  $(\cdot, \cdot)$  symmetrischer Operator, der eine selbstadjungierte Fortsetzung  $A^* = A^* \supset L$  besitzt \*).*

a) *Dann kann man den Kern  $K$  nach Eigenkernen  $\Theta(x, y; \lambda)$ ,  $\Theta(\cdot, \cdot; \lambda) \in \mathcal{D}'^m(\Omega_n \times \Omega_n)$  folgendermassen entwickeln:*

$$(4) \quad K(x, y) = \int_{\text{sp } A} \Theta(x, y; \lambda) d\mu(\lambda),$$

wo  $\mu$  ein positives Mass auf  $\text{sp } A$  bedeutet.

Die Spektralfunktion  $\Theta(x, y; \lambda)$  ist Eigendistribution von  $L$ , d. h. es gilt

$$(5) \quad L'_x \Theta(x, y; \lambda) = \overline{L'_y} \Theta(x, y; \lambda) = \lambda \Theta(x, y; \lambda)$$

$$(L'u, v)_0 \stackrel{u, v}{=} (u, Lv)_0 \overline{L'u} \stackrel{\text{df}}{=} \overline{Lu} \quad \text{wo} \quad \overline{u}(x) = \overline{u(x)}.$$

b) *Falls  $L$  hypoelliptisch ist, d. h. wenn jede Eigenlösung im Sinne der Distributionentheorie eine (hinreichend reguläre — sagen wir in der Klasse  $C^m$ ) Lösung im klassischen Sinne ist, dann ist  $\Theta(\cdot, \cdot; \lambda)$  regulär:*

$$\Theta(\cdot, y; \lambda) \in C^m(\Omega_n), \quad \Theta(x, \cdot; \lambda) \in C^m(\Omega_n)$$

und die  $\Theta(\cdot, \cdot; \lambda)$  sind Eigenfunktionen, d. h. die Gleichungen (5) gelten im klassischen Sinne.

In diesem Falle hat man

$$(6) \quad \Theta(u, v; \lambda) = (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) = \int_{\Omega_n \times \Omega_n} \Theta(x, y; \lambda) u(x) \overline{v(y)} dx dy$$

identisch für  $u, v \in C_0^\infty(\Omega_n)$ .

Es gilt trivialerweise der entsprechende Umkehrungssatz.

Durch Spezialisierung erhalten wir folgende wichtigen Korolläre:

**KOROLLAR 1.** *Der Satz von Berezanskij: man soll den Kern  $K$  stetig und den Operator  $L$  als elliptischen Differentialoperator voraussetzen.*

**KOROLLAR 2.** *Verallgemeinerung eines Satzes von Gårding über die Differenzierbarkeit der Spektralfunktion [9]: man soll  $K(x, y) = \delta(x - y)$  („Diracsche  $\delta$ “) und  $L$  — elliptisch voraussetzen.*

\*)  $L$  braucht kein Differentialoperator zu sein.

## Beweis des Hauptsatzes

ad a) Aus dem Gårdingschen Spektralsatz wissen wir, dass  $\hat{u}(\lambda), \hat{v}(\lambda)$  vektorielle Distributionen der Ordnung  $Q \leq n + m$  sind.

Aus der Stetigkeit des skalaren Produktes  $(\hat{u}(\lambda), \hat{v}(\lambda); \lambda)$  (Schwartzsche Ungleichung) folgt also, dass  $(\hat{u}(\lambda), \hat{v}(\lambda); \lambda)$  eine stetige hermitesche positive Form  $\Theta(u, v; \lambda)$  auf  $\mathcal{D}^Q(\Omega_n) \times \mathcal{D}^Q(\Omega_n)$  ist.

$$(7) \quad \Theta(u, v; \lambda) \stackrel{\text{def}}{=} (\hat{u}(\lambda), (\hat{v}\lambda); \lambda).$$

Es existiert also ein solcher hermitescher (positiver) Kern

$$\Theta_\lambda \in \mathcal{D}'^{\Theta}(\Omega_n \times \Omega_n),$$

dass

$$(8) \quad (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) = \Theta(u, v; \lambda) = \langle \Theta_\lambda u, v \rangle$$

identisch für  $u, v \in C_0^\infty$  gilt.

Da

$$\begin{aligned} (u, Lv) &= (Lu, v) = \int \Theta(Lu, v; \lambda) d\mu(\lambda) = \int \Theta(u, Lv; \lambda) d\mu(\lambda) = \\ &= \int \lambda (\hat{u}(\lambda), \hat{v}(\lambda); \lambda) d\mu(\lambda) = \int \lambda \Theta(u, v; \lambda) d\mu(\lambda) \end{aligned}$$

haben wir auf  $\text{sp } A$  fast  $\mu$ -überall.

$$(9) \quad \Theta(Lu, v; \lambda) = \Theta(u, Lv; \lambda) = \lambda \Theta(u, v; \lambda)$$

identisch für  $u, v \in C_0^\infty(\Omega_n)$ .

Aus (8) und (9) haben wir

$$(10) \quad \langle \Theta_\lambda Lu, v \rangle \equiv \langle \Theta_\lambda u, Lv \rangle \equiv \langle \lambda \Theta_\lambda u, v \rangle; \quad u, v \in C_0^\infty.$$

Die Formeln (2), (8) und (10) geben die Behauptung

$$(u, v) = \langle Ku, v \rangle \equiv \int_{\text{sp } A} \langle \Theta_\lambda u, v \rangle d\mu(\lambda).$$

ad b) Es folgt aus (10), dass

$$\langle \Theta_\lambda u, (L - \lambda)v \rangle \equiv 0 \quad u, v \in C_0^\infty(\Omega_n).$$

Da wir jetzt Hypoelliptizität von  $L$  voraussetzen, es folgt daraus, dass

$$\Theta_\lambda u \in C^m(\Omega_n)$$

und dass

$$(L' - \lambda)\Theta_\lambda u \equiv 0$$

identisch für  $u \in C_0^\infty(\Omega_n)$ , wo  $L'$  der in bezug auf

$$(u, v)_0 \stackrel{\text{def}}{=} \int_{\Omega_n} u(x) \overline{v(x)} dx$$

adjungierte von  $L$  ist, d. h.

$$(u, Lv)_0 \equiv (L'u, v), \quad u, v \in C_0^\infty.$$

Wir können also schreiben

$$(11) \quad \langle \Theta_\lambda u, v \rangle = \int_{\Omega_n} \Theta_\lambda u(y) \overline{v(y)} dy.$$

$\Theta_\lambda: u \rightarrow \Theta_\lambda u$  ist definitionsgemäss eine stetige lineare Abbildung von  $\mathcal{D}$  in  $\mathcal{D}'^m$ , aber — wie wir oben gezeigt haben —  $\Theta_\lambda u \in C^m(\Omega_n)$ . Wir sehen also, dass der Graph von  $\Theta_\lambda: \mathcal{D} \rightarrow \mathcal{E}^m$  \*) abgeschlossen ist (aus  $u_\nu \xrightarrow{\mathcal{D}} 0$  und  $\Theta_\lambda u_\nu \rightarrow w$  folgt:  $w = 0$ ).

Nach der allgemeinen Fassung des Satzes von abgeschlossenen Graphen [3] ist die Abbildung  $\Theta_\lambda: \mathcal{D} \rightarrow \mathcal{E}^m$  stetig. Insbesondere ist die Abbildung  $\mathcal{D} \ni u \rightarrow \Theta_\lambda u(y) \in C^1$  eine Distribution. Wir können also schreiben

$$(12) \quad \overline{\langle \Theta(\cdot, y; \lambda), u \rangle} \stackrel{\text{df}}{=} \Theta_\lambda u(y)$$

identisch für  $u \in C_0^\infty(\Omega_n)$ .

Es folgt aber aus der linken Seite der Formel (10), dass

$$\langle \Theta(\cdot, y; \lambda), Lu \rangle = \lambda \langle \Theta(\cdot, y; \lambda), u \rangle$$

identisch für  $u \in C_0^\infty$ .

Daher folgt aus der Hypoelliptizität von  $L$ , dass

$$\begin{aligned} \Theta(\cdot, y; \lambda) &\in C^m(\Omega_n) \\ L_y \Theta(\cdot, y; \lambda) &= \lambda \Theta(\cdot, y; \lambda). \end{aligned}$$

Aus der hermitescher Symmetrie von  $\Theta_\lambda$  folgt es, dass

$$(13) \quad L_y \Theta(x, \cdot; \lambda) = \overline{\lambda \Theta(x, \cdot; \bar{\lambda})} \in C^m(\Omega_n)$$

was eben

$$\overline{L_y \Theta}(x, \cdot; \lambda) = \lambda \Theta(x, \cdot; \lambda)$$

bedeutet.

Aus (8), (11), (12) und (13) folgt (6)

$$0 \leq \Theta(u, u; \lambda) = \int_{\Omega_n \times \Omega_n} \Theta(x, y; \lambda) u(x) \overline{u(y)} dx dy$$

und

$$\langle Ku, v \rangle = \int_{\text{sp} A} \int_{\Omega_1 \times \Omega_n} \Theta(x, y; \lambda) u(x) \overline{v(y)} dx dy d\mu(\lambda),$$

---

\*)  $\mathcal{E}^m = \mathcal{E}^m(\Omega_n)$  ist die Menge  $C^m(\Omega_n)$  mit der Schwartzschen Topologie, d. h.  $u_\nu \xrightarrow{\mathcal{E}^m} 0$ , wenn  $u_\nu \rightarrow 0$  gleichmässig mit allen Ableitungen bis zur Ordnung  $m$  auf jeder kompakten Untermenge.  $\mathcal{E} \stackrel{\text{df}}{=} \mathcal{E}^\infty$ .

was eben die Spektraldarstellung des Kernes  $K$  bedeutet und wofür man eben die Schreibweise

$$K(x, y) = \int_{\text{sp.} A} \Theta(x, y; \lambda) d\mu(\lambda)$$

verwendet. Der Beweis des Hauptsatzes ist somit vollständig erbracht.

Bemerkung: Bei dem Beweis der Differenzierbarkeit der Spektralfunktion  $\Theta(x, y, \lambda)$  haben wir keine Existenz der Fundamentallösung von  $L$  vorausgesetzt.

### Einige Verallgemeinerungen

Wie vor Kurzem Browder [2] bemerkt hat, gilt der vollständige Spektralsatz in sinngemässer Formulierung auch für einen subnormalen Operator  $L$ , d. h. für einen solchen Operator, der in einem Raume  $\tilde{\mathfrak{H}} \supset \mathfrak{H}$  ( $\mathfrak{H}$  ist dabei ein abgeschlossener Unterraum von  $\mathfrak{H}$ ) eine normale Fortsetzung  $A$  besitzt. (Alle symmetrische Operatoren sind bekanntlich subnormal).

Der Hauptsatz gilt also für beliebige subnormale Operatoren  $L$ . Man kann selbstverständlich statt skalaren Funktionen  $u, v$  die  $N$ -Komponenten-Vektorfelder  $u, v \in C_0^{\infty, N}(\Omega_n)$  auf  $\Omega_n$  nehmen. Man kann also den Hauptsatz für ein System der  $N$ -skalaren Operatoren aussprechen.

### Entwicklung der positiv definiter invarianter Kerne auf homogenen Räumen

Man kann dem Hauptsatz auch eine andere wichtige Wendung geben. Wie L. Ehrenpreis [5], [6] gezeigt hat, kann man die Distributionstheorie und — was für uns bedeutend ist — auch die Theorie der Kerne für den Doppelnebenklassenraum („double coset space“)  $L \backslash G / K$  einer separablen Lieschen Gruppe  $G$  ( $K$  und  $L$  sind kompakte Untergruppen von  $G$ ) entwickeln. Wir haben also hiermit den folgenden Satz bewiesen.

**SATZ 2.** *Der Hauptsatz gilt mutatis mutandis, wenn man  $\Omega_n = L \backslash G / K$  setzt.*

Der Satz 2 lässt also wichtige Anwendungen in der Theorie der allgemeinen sphärischen Funktionen von Gelfand [7] und M. G. Krein [10] zu. Diesem letzten Thema wird eine andere Abhandlung gewidmet.

Die Anregung zu dieser Abhandlung bekam ich von Professor Lars Gårding, während meines Aufenthaltes in Lund, wofür ich ihm hier meinen herzlichsten Dank aussprechen möchte.

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# On the Derivative of Bounded Univalent Functions

by

J. KRZYŻ

*Presented by K. KURATOWSKI on January 18, 1958*

In a recent paper [1] W. Janowski, using the variational methods of Z. Charzyński [5], obtained the precise bounds for the modulus of the derivative of a function  $f(z)$  univalent and bounded in the unit circle.

In this paper, we give an elementary and simple derivation of the precise lower bound for  $|f'(z)|$  valid for all  $|z| < 1$ , and of the precise upper bound for  $|f'(z)|$  valid for  $|z| < r(M)$ , where  $r = r(M)$  depends on  $M \geq \sup_{|z| < 1} |f(z)|$ . Furthermore, we announce some results, the proofs of which will soon be published in the Annales UMCS.

Let  $(F_M)$  denote the class of functions  $f(z) = z + a_2 z^2 + \dots$  regular and univalent for  $|z| < 1$ , and such that  $\sup_{|z| < 1} |f(z)| \leq M$ . Clearly,  $(F_M)$  is void if  $M < 1$  and  $(F_1)$  contains the function  $f(z) \equiv z$  only. Let  $(G_M)$  denote the class of functions of the form

$$(1) \quad g(z) = f(z) [1 + e^{i\alpha} M^{-1} f(z)]^{-2}$$

with real, arbitrary  $\alpha$  and  $f(z) \in (F_M)$ .

It is easy to see that every (non-void) class  $(G_M)$  contains the Koebe function  $f_0(z) = z(1-z)^{-2}$ .

For  $M = 1$  this assertion is trivial, for  $M > 1$  put

$$(2) \quad w = f_M(z) = M \left[ \frac{z}{M(1-z)^2} + \frac{1}{2} - \sqrt{\frac{z}{M(1-z)^2} + \frac{1}{4}} \right] \left[ \frac{z}{M(1-z)^2} \right]^{-1}$$

(we take that branch of radical, which takes the value  $1/2$  for  $z = 0$ ). Obviously  $f_M(z)$  is the well known Pick function, which represents the unit circle  $|z| < 1$  on the circle  $|w| < M$  slit along the segment  $-M \leq w \leq -M[2M - 1 - 2\sqrt{M(M-1)}]$ . Since  $f_M(0) = 0$ ,  $f'_M(0) = 1$ ,  $f_M(z) \in (F_M)$ . Now, an easy calculation gives  $z(1-z)^{-2} \equiv f_M(z)[1 - M^{-1}f_M(z)]^{-2}$  and this proves our assertion. Since every class  $(G_M)$  is composed of uni-

valent functions and contains the Koebe function, the precise bounds for the moduli of function, its derivative and its logarithmic derivative in  $(G_M)$  are the same as in  $(F_\infty)$ . In particular,

$$|z|(1+|z|)^{-2} \leq |f(z)| |1 + e^{ia} M^{-1} f(z)|^{-2} \leq |z|(1-|z|)^{-2}.$$

Since  $a$  is arbitrary, the right hand side inequality gives

$$(3) \quad |f(z)| (1 - M^{-1} |f(z)|)^{-2} \leq |z| (1 - |z|)^{-2}$$

and the left hand side:

$$(3a) \quad |z| (1 + |z|)^{-2} \leq |f(z)| (1 + M^{-1} |f(z)|)^{-2}.$$

The inequalities (3) and (3a) imply

$$(4) \quad -f_M(-|z|) \leq |f(z)| \leq f_M(|z|).$$

This is a well known result obtained by G. Pick [4]. We have

$$\left| \frac{g'(z)}{g(z)} \right| = \left| \frac{f'(z)}{f(z)} \right| \left| \frac{1 - e^{ia} M^{-1} f(z)}{1 + e^{ia} M^{-1} f(z)} \right|,$$

and the well known inequalities for logarithmic derivative, ([3], pp. 92, 93) give

$$(5) \quad \frac{1}{|z|} \frac{1 - |z|}{1 + |z|} \leq \left| \frac{f'}{f} \right| \frac{1 - e^{ia} M^{-1} f}{1 + e^{ia} M^{-1} f} \leq \frac{1}{|z|} \frac{1 + |z|}{1 - |z|}.$$

Since  $a$  is arbitrary, we have

$$(6) \quad \frac{M}{|z|} \frac{1 - |z|}{1 + |z|} \frac{M^{-1} |f| (1 + M^{-1} |f|)}{1 - M^{-1} |f|} \leq |f'|.$$

A lower bound for  $|f'(z)|$  which does not depend on  $|f(z)|$  ( $|z|$  being fixed) can be obtained by putting in (6) instead of  $M^{-1} |f|$  the least possible value, i. e.  $-M^{-1} f_M(-|z|)$ . This is obvious in view of the fact that  $u(1+u)(1-u)^{-1}$  is a strictly increasing function of  $u$  for  $0 < u < 1$ .

An easy calculation shows that, for  $f = f_M$  and  $z = |z|$ , an equality takes place in (6); thus, the lower bound obtained is the best possible one. Quite similarly, the right hand side inequality in (5) gives

$$(7) \quad |f'| \leq \frac{M}{|z|} \frac{1 + |z|}{1 - |z|} \frac{M^{-1} |f| (1 - M^{-1} |f|)}{1 - M^{-1} |f|}.$$

The function  $u(1-u)(1+u)^{-1}$  is strictly increasing for  $0 < u < \sqrt{2}-1$  and this means that, for  $z$  such that  $M^{-1} f_M(|z|) < \sqrt{2}-1$ , i. e. for

$$|z| \leq 1 - M^{-1} (1 + \sqrt{2})^{-1} \left[ (1 + 2M(1 + \sqrt{2}))^{1/2} - 1 \right],$$

an upper bound for  $|f'|$  can be obtained by putting in (7)  $f_M(|z|)$  instead of  $|f|$ , with the sign of equality for  $f = f_M$  and  $z = |z|$ .

Since  $\sup_{0 < u < 1} u(1-u)(1+u)^{-1} = 3-2\sqrt{2} = 0.17\dots$ , we see that

$$(7a) \quad |f'(z)| \leq \frac{M}{|z|} \frac{1+|z|}{1-|z|} (3-2\sqrt{2}).$$

The inequality (7a) means that  $f'(z) = o\left(\frac{1}{1-|z|}\right)$ .

We can, however, easily prove that, in this result, "0" may be replaced by "o" and that the thus improved result applies to a more general class of functions. In fact, the following theorems hold:

**THEOREM 1.** *If the function  $f(z)$  is regular in the unit circle  $|z| < 1$  and if the Riemann surface being the map of  $|z| < 1$  by  $f(z)$  is of finite area, then  $f'(z) = o(1-|z|)^{-1}$ .*

**THEOREM 2.** *The inner radius  $r(D, P)$  of a simply connected Riemann surface  $D$  of finite area at the point  $P \in D$  tends to zero as  $P$  approaches the ideal boundary of  $D$ .*

The proofs are contained in [2].

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# An Example of a Non-normal Completely Regular Space

by

S. MRÓWKA

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The first example of a non-normal completely regular space was given by A. Tychonoff [1]; this example is constructed with the use of transfinite ordinals. P. Alexandroff raised the problem of constructing such a space without the use of transfinite ordinals. This problem was solved by Nemytzki\*). The space constructed by Nemytzki is very simple, but the proof that it is completely regular is rather complicated; it may be found in [2].

In this paper we give a very simple example of non-normal completely regular space; our example is a modification of Nemytzki's example.

Let  $X$  be the set of all points  $p = (x, y)$  of the Euclidean plane, for which  $y = 0$ , or  $y > 0$  and  $x$  and  $y$  are rational. (Thus  $X$  consists of all points of the plane lying on the line  $y = 0$  and of all points lying above this line and having rational co-ordinates). Neighbourhoods of points  $p$  lying on the line  $y = 0$  are of the form  $\{p\} \cup K_p$ , where  $K_p$  is any open disc which is tangential to the line  $y = 0$  at the point  $p$ ; all remaining points of  $X$  are isolated. Clearly, each set of the form  $\{p\} \cup K_p$  ( $p$  lies on the line  $y = 0$ ) is both closed and open in  $X$ ; thus,  $X$  has a basis of closed-open neighbourhoods; consequently,  $X$  is completely regular. On the other hand, it is plain that  $X$  is Hausdorff.

Denote by  $N$  and  $M$  the sets of all irrational and rational points of the line  $y = 0$ , respectively. The sets  $N$  and  $M$  are disjoint and closed in  $X$ . Let  $G$  be any open set containing  $N$ . Denote by  $N_n$  the set of all points  $p$  of  $N$  for which there is a neighbourhood  $\{p\} \cup K_p \subset G$ , where  $K_p$  is of diameter  $\geq 1/n$ . Since  $N = \bigcup_n N_n$  and the set  $N$  is a topologically complete space (with respect to the usual topology of the line), then,

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\*) For references and other informations relating to this topic see [2].

by the Baire theorem on category, there is an index  $n_0$  such that the set  $N_{n_0}$  is dense in some interval  $(a, b)$ . Clearly,  $G$  contains all points  $p = (x, y) \in X$  for which  $a \leq x \leq b$ ;  $0 \leq y \leq 1/n$ . It follows that, if  $p$  is any point of  $M$  lying in the interval  $(a, b)$ , then each neighbourhood of  $p$  intersects  $G$ . Consequently, the sets  $N$  and  $M$  have no disjoint neighbourhoods and the space  $X$  is non-normal.

1. Note the following strange properties of the space  $X$ :

- (1) The space  $X$  is non-paracompact,
- (2) The space  $X$  is the union of two paracompact subsets.

Denote as  $X_1$  the set of all points of  $X$  lying on the line  $y = 0$  and let  $X_2 = X \setminus X_1$ . Then  $X = X_1 \cup X_2$  and the sets  $X_1$  and  $X_2$  are paracompact (they are discrete in their relative topology).

(3) The space  $X$  is the union of an infinite sequence of closed paracompact subsets.

In fact, the set  $X_1$  is closed in  $X$  and the set  $X_2$  is countable.

It follows, that *paracompactness and normality are not finitely additive and are not countably additive with respect to closed subsets*. On the other hand, it is known that paracompactness and normality are finitely additive with respect to closed subsets [3].

It may also be noticed that the space  $X$  contains an everywhere dense countable subset and satisfies the first axiom of countability, but, being non-normal, does not satisfy the second.

2. It may be easily shown that the space  $X$  is a  $Q$ -space [4]. In fact, T. Shirota has proved the following theorem ([5], Theorem 6):

*The following conditions on a completely regular space  $Y$  are equivalent:*

- (i) *if a completely regular space  $X$  is the domain of a continuous one-to-one mapping onto  $Y$ , then  $X$  is a  $Q$ -space,*
- (ii) *every subset of  $Y$  is a  $Q$ -space\*).*

Denote by  $Y$  the space consisting of all points of  $X$ , but with the ordinary Euclidean topology. Then, the identity mapping is a continuous one-to-one mapping of  $X$  onto  $Y$ ; indeed, each neighbourhood of a point of  $Y$  is contained in some Euclidean neighbourhood of this point. On the other hand, we infer from the Hewitt theorem, stating that a completely regular space which satisfies the second axiom of countability is a  $Q$ -space, ([4], Theorem 53), that the space  $Y$  satisfies the condition (ii) of the Shirota theorem. Thus  $X$  is a  $Q$ -space.

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\*) Shirota uses the term "*e*-complete space" instead of our " $Q$ -space". But Theorem 2 of the cited paper of Shirota states that a completely regular space is *e*-complete if, and only if, it is a  $Q$ -space.

3. Let us re-state certain properties of the space  $X$ :

*The space  $X$  is:*

- 1) *completely regular;*
- 2) *having a basis of closed-open neighbourhoods;*
- 3) *non-normal;*
- 4) *having an everywhere dense countable subset and satisfying the first axiom of countability, but not satisfying the second,*
- 5) *a  $Q$ -space.*

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# Sur un problème concernant le nombre de diviseurs d'un nombre naturel

par

A. SCHINZEL

*Présenté par W. SIERPIŃSKI le 18 janvier 1958*

S. Golomb a démontré [1], qu'ils existent pour tout  $k$  naturel des nombres  $c$  tels, que  $\mu(c+1) = \dots = \mu(c+k) = 0$  où  $\mu$  désigne la fonction de Möbius, et a posé la question de la vérité d'un théorème analogue pour les fonctions  $\theta$ ,  $\varphi$  et  $\sigma$ .

Dans un travail antérieur [2], l'hypothèse H a été énoncé, qui suppose que si  $f_i(x)$  ( $i = 1, 2, \dots, k$ ) est une suite finie de polynômes en  $x$  aux coefficients entiers, où les coefficients des puissances les plus grandes de  $x$  sont positifs, irréductibles et tels, qu'il n'existe aucun nombre naturel  $>1$  qui divise le nombre  $P(x) = f_1(x)f_2(x) \dots f_k(x)$ , quel que soit l'entier  $x$ , alors il existe une infinité de nombres naturels  $x$  tels, que chacun de nombres  $f_i(x)$  ( $i = 1, 2, \dots, k$ ) est premier.

Le but de la présente note est de déduire de l'hypothèse H la proposition P suivante:

PROPOSITION P.  $\theta(n)$  désignant le nombre de diviseurs naturels du nombre  $n$ , il existe pour tout nombre naturel  $k > 1$  une infinité de nombres naturels  $n$  tels que

$$\theta(n+1) = \theta(n+2) = \dots = \theta(n+k).$$

Démonstration de l'implication  $H \rightarrow P$ .

$p_i$  désignant le  $i$ -ème nombre premier, il existe pour le nombre naturel donné  $k > 1$  un nombre naturel  $r$  tel, que  $p_r \leq k < p_{r+1}$ .

Or, soit  $M = [\theta(1), \theta(2), \dots, \theta(k)]$  et soit  $\alpha_i = \frac{M}{\theta(i)} - 1$ .

En vertu du théorème chinois sur les restes il existe un entier  $x_0 > 0$  tel que

$$x_0 \equiv 0 \pmod{k! \cdot p_1 \cdot \dots \cdot p_r}$$

$$x_0 \equiv p_{r+i}^{\alpha_i} - i \pmod{p_{r+i}^{\alpha_i+1}} \quad (i = 1, 2, \dots, k).$$

Posons

$$f_i(x) = \frac{k! \prod_{j=1}^k p_{r+j}^{\alpha_j}}{i p_{r+i}^{\alpha_i}} x + \frac{x_0 + i}{i p_{r+i}^{\alpha_i}}, \quad P(x) = \prod_{i=1}^k f_i(x).$$

Vu la définition de  $x_0$ , les coefficients des binômes  $f_i(x)$  ( $i = 1, 2, \dots, k$ ) sont naturels et on a  $\Delta^k P(x) = k! \left( k! \prod_{i=1}^k p_{r+i}^{\alpha_i} \right)^{k-1}$ . Tout nombre naturel  $m$  qui divise  $P(x)$  pour tout  $x$  entier est donc un diviseur de  $k! \left( k! \prod_{i=1}^k p_{r+i}^{\alpha_i} \right)^{k-1}$ .

D'autre part

$$P(0) = \prod_{i=1}^k \frac{x_0 + i}{i p_{r+i}^{\alpha_i}}.$$

Mais

$$\frac{x_0 + i}{i} \equiv 1 \left( \text{mod } \frac{k!}{i} p_1 \dots p_r \right),$$

done  $(P(0), k!) = 1$ .

Or  $\frac{x_0 + i}{i p_{r+i}^{\alpha_i}} \equiv 1 \pmod{p_{r+i}}$  et enfin pour tous  $j \neq i$ , tels que  $\alpha_j > 0$  on a

$$\left( p_{r+j}, \frac{x_0 + i}{i p_{r+i}^{\alpha_i}} \right) | (x_0 + j, x_0 + i) | |j - i| < p_{r+j},$$

d'où

$$\left( P(0), \prod_{i=1}^k p_{r+i}^{\alpha_i} \right) = 1.$$

Il en résulte, qu'il n'existe aucun nombre naturel  $> 1$  qui divise  $P(x)$  pour tout  $x$  entier et, d'après l'hypothèse H il existe une infinité de nombres naturels  $x$  pour lesquels les nombres  $f_i(x)$  ( $i = 1, 2, \dots, k$ ) sont premiers et  $> p_{r+k}$ . En posant

$$n = k! \prod_{i=1}^k p_{r+i}^{\alpha_i} x + x_0$$

on a pour  $i \leq k$

$$\theta(n+i) = \theta(i) \theta(p_{r+i}^{\alpha_i}) \theta(f_i(x)) = \theta(i) \cdot \frac{M}{\theta(i)} \cdot 2 = 2M.$$

En appliquant la même méthode on peut déduire de l'hypothèse H une proposition analogue pour la fonction  $\nu(n)$  (nombre de diviseurs premiers du nombre  $n$ ).

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# Solution générale d'un problème de Sikorski

par

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Présenté par K. KURATOWSKI le 20 janvier 1958

R. Sikorski a posé le problème suivant [3] \*):

„Des fonctions réelles continues  $f_1$  et  $f_2$  étant définies sur le segment  $0 \leq x \leq 1$  et y ayant les propriétés suivantes:

$$(1) \quad f_i(0) = 0, \quad 0 \leq f_i(x) \leq 1 \quad \text{et} \quad f_i(1) = 1 \quad \text{pour} \quad i = 1 \text{ et } 2,$$

$$(2) \quad \text{ni } f_1, \text{ ni } f_2 \text{ n'est constante sur aucun intervalle,}$$

est-ce que toute composante de l'ensemble  $\bigcup_{(x,y)} \{f_1(x) = f_2(y)\}$  est localement connexe?"

Désignons cet ensemble par  $A_2$ .

J. S. Lipiński a démontré davantage [2], à savoir que toute composante de l'ensemble  $A_2$  (qui est un continu,  $A_2$  étant évidemment compact) est héréditairement localement connexe (c'est-à-dire ne contient que des continus localement connexes). Quant au cas de  $n \geq 2$  fonctions, la connexité locale de toute composante de l'ensemble

$$(3) \quad A_n = \bigcup_{(x_2, x_2, \dots, x_n)} \{f_1(x_1) = f_2(x_2) = \dots = f_n(x_n)\}$$

a été établie par R. Sikorski et K. Zarankiewicz ([4] et [5]), mais sous des hypothèses supplémentaires.

Sans faire intervenir ces résultats, le théorème qui va suivre en est une généralisation dans deux sens: il se passe de l'hypothèse (1) et sa thèse précise davantage la structure des composantes de l'ensemble  $A_n$  en question par la mise au point de celle de cet ensemble tout entier.

Je dis qu'un ensemble  $E$  est d'ordre  $\leq n$  au sens fort au point  $p$  lorsqu'il existe, pour un  $m > 0$ , une famille de domaines ouverts ([1], I, p. 43)  $G_\varepsilon$  où  $0 < \varepsilon < m$ , tels que  $\bar{G}_{\varepsilon'} \subset G_{\varepsilon''}$  pour  $\varepsilon' < \varepsilon''$  (famille strictement monotone au sens de [1], I, p. 155), que  $(p) = \bigcap_{\varepsilon \rightarrow 0} \bar{G}_\varepsilon$  et que  $\text{Fr}(\bar{G}_\varepsilon) \cdot E$

\*) Pour l'origine, l'ordre d'idées et la littérature du problème, voir [2], p. 1019.

se compose de  $n$  points au plus pour chacune des  $2^{\aleph_0}$  valeurs de l'indice  $\varepsilon$  en question.

Ainsi,  $I$  désignant le segment  $0 \leq x \leq 1$  de l'axe d'abscisses, le point  $p = (0, 0)$  par exemple est d'ordre 1, mais pas au sens fort, dans la courbe  $C$ , qui résulte de  $I$  en y remplaçant les intervalles contigus à l'ensemble des points de la forme  $(k^{-1}, 0)$  pour  $k = 1, 2, \dots$  (la suite harmonique) par les circonférences ayant ces intervalles pour diamètres; de même, tout point  $p$  de la courbe triangulaire de Sierpiński ([1], II, p. 203) est d'ordre  $\leq 4$ , mais il est d'ordre infini au sens fort.

Soit  $f$  une fonction réelle, continue sur  $I$  et qui n'y est constante dans aucun sous-intervalle. Fixons un point  $a \in I$  et posons

$$(4) \quad \begin{aligned} g_a &= \max_{0 \leq x \leq a} |f(x) - f(a)|, & d_a &= \max_{a \leq x \leq 1} |f(x) - f(a)|, \\ m_a &= \begin{cases} d_a & \text{lorsque } a = 0, \\ \min(g_a, d_a) & \text{lorsque } 0 < a < 1, \\ g_a & \text{lorsque } a = 1. \end{cases} \end{aligned}$$

Etant donné un ensemble  $J \subset I$ , convenons d'entendre par  $\text{Fr}(J)$  sa frontière relative à  $I$  (c'est-à-dire l'ensemble  $\bar{J} \cdot I - J$ ).

On aura à appliquer les propriétés suivantes:

(i) On a  $m_a > 0$  pour tout  $a \in I$ .

En effet, le contraire veut dire d'après (4) que  $f(x)$  prend la valeur constante  $f(a)$  pour  $0 \leq x \leq a$  ou pour  $a \leq x \leq 1$  respectivement.

(ii) Il existe pour tout  $\varepsilon < m_a$  un intervalle (ouvert relativement à  $I$ ) le plus grand  $J_\varepsilon \subset I$ , tel que  $a \in J_\varepsilon$  et

$$(5) \quad |f(x) - f(a)| < \varepsilon \quad \text{pour tout } x \in J_\varepsilon;$$

cet intervalle est unique et

$$(6) \quad \text{Fr}(\bar{J}_\varepsilon) \text{ ne contient ni } 0, \text{ ni } 1.$$

Soit, en effet,  $J_\varepsilon$  la réunion de tous les intervalles  $J \subset I$  ouverts dans  $I$  et tels que  $a \in J$  et  $|f(x) - f(a)| < \varepsilon$  pour tout  $x \in J$ ; de tels  $J$  existent par suite de la continuité de  $f(x)$ . Si un point de  $\text{Fr}(\bar{J}_\varepsilon)$  se trouvait par exemple au point 0, on aurait  $|f(0) - f(a)| \leq \varepsilon$  en vertu de la dernière inégalité et comme  $\varepsilon < m_a$ , il viendrait  $\varepsilon < |f(x) - f(a)| < m_a$  d'après (4) pour un  $x$  compris entre 0 et 1, donc pour un  $x \in J_\varepsilon$ , contrairement à la définition de  $J_\varepsilon$ .

(iii) Si  $x' \in \text{Fr}(J_\varepsilon)$ , on a  $|f(x') - f(a)| = \varepsilon$ .

En effet, la continuité de  $f(x)$  entraîne d'après (5) que  $|f(x') - f(a)| \leq \varepsilon$  et, qu'en admettant le signe  $<$ , il existerait d'après (6) un intervalle  $J'$  ouvert relativement à  $I$ , contenant le point  $x'$  et dont tous les points  $x$  satisferaient encore à l'inégalité  $|f(x) - f(a)| < \varepsilon$ , qui se trouverait ainsi satisfaite dans l'intervalle  $J_\varepsilon + J'$ , plus grand que  $J_\varepsilon$ .

(iv) La famille indénombrable des segments  $\bar{J}_\varepsilon$  pour  $0 < \varepsilon < m_a$  est strictement monotone.

En effet,  $0 < \varepsilon' < \varepsilon''$  entraîne  $\bar{J}_{\varepsilon'} \subset J_{\varepsilon''}$ , car on a trivialement  $J_{\varepsilon'} \subset J_{\varepsilon''}$  et en supposant l'existence d'un  $x' \in \text{Fr}(\bar{J}_{\varepsilon'}) \cdot \text{Fr}(\bar{J}_{\varepsilon''})$ , le nombre  $|f(x') - f(a)|$  serait égal en vertu de (iii) à deux nombres positifs inégaux  $\varepsilon'$  et  $\varepsilon''$  à la fois.

(v) On a  $\delta(J_\varepsilon) \rightarrow 0$  pour  $\varepsilon \rightarrow 0$  ( $\delta$  désignant le diamètre); en d'autres termes,  $(a) = \prod_{\varepsilon \rightarrow 0} J_\varepsilon$ .

En effet, si cette partie commune des  $J_\varepsilon$  contenait un intervalle, on y aurait  $|f(x) - f(a)| < \varepsilon$  pour tout  $\varepsilon > 0$ , c'est-à-dire  $f(x) = f(a)$ , contrairement à l'hypothèse que  $f(x)$  n'est constante dans aucun intervalle.

**THÉORÈME.** *Si les fonctions  $f_1, f_2, \dots, f_n$  aux valeurs réelles sont définies sur  $I$ , continues sur ce segment et ne sont constantes dans aucun intervalle, l'ensemble  $A_n$  est d'ordre  $\leq 2^n$  au sens fort en chacun de ses points.*

**Démonstration.** Soit  $p = (a_1, a_2, \dots, a_n)$  un point quelconque de  $A_n$ . On a donc par définition

$$(7) \quad f_1(a_1) = f_2(a_2) = \dots = f_n(a_n),$$

où  $a_i \in I$  pour tout  $i = 1, 2, \dots, n$ . Les symboles  $g_{i_{a_i}}, d_{i_{a_i}}, m_{a_i}$  et  $J_{\varepsilon_i}$  relatif à  $f_i$  étant définis comme  $g_a, d_a, m_a$  et  $J_\varepsilon$  l'ont été pour  $f$ , on a  $m_{a_i} > 0$  en vertu de (i). Considérons les intérieurs des parallélépipèdes à  $n$  dimensions

$$(8) \quad G_\varepsilon = J_{\varepsilon_1} \times J_{\varepsilon_2} \times \dots \times J_{\varepsilon_n}, \quad \text{où} \quad 0 < \varepsilon < \min_{i=1,2,\dots,n} m_{a_i}.$$

Les  $G_\varepsilon$  sont donc des domaines ouverts de l'espace  $E^n$  et contiennent le point  $p$ . Leur famille est définie pour tout  $\varepsilon$  d'un voisinage positif du point 0, elle est strictement monotone en vertu de (iv) et on a  $(p) = \prod_{\varepsilon \rightarrow 0} G_\varepsilon$  en vertu de (v).

Reste à évaluer le nombre des points de  $\text{Fr}(\bar{G}) \cdot A_n$ . Il suffit de montrer que cet ensemble est contenu dans le suivant:

$$(9) \quad S_\varepsilon = \text{Fr}(J_{\varepsilon_1}) \times \text{Fr}(J_{\varepsilon_2}) \times \dots \times \text{Fr}(J_{\varepsilon_n}),$$

qui l'est dans celui des  $2^n$  sommets de  $G_\varepsilon$  (les dernières  $n$  frontières étant entendues, évidemment, dans les espaces-facteurs respectifs). Or, si  $p' = (x'_1, x'_2, \dots, x'_n) \in \text{Fr}(\bar{G}) \cdot A_n$ , on a  $x'_j \in \text{Fr}(J_{\varepsilon_j})$  pour une coordonnée  $x'_j$  de  $p'$ . Il en résulte en vertu de (7) et (iii) que

$$|f_j(x'_j) - f_j(a_j)| = |f_j(x'_j) - f_j(a_j)| = \varepsilon \quad \text{pour tout} \quad i = 1, 2, \dots, n$$

et on peut y remplacer  $j$  par  $i$ , puisque  $p' \in A_n$  par hypothèse, d'où  $f_1(x'_1) = f_2(x'_2) = \dots = f_n(x'_n)$ . On a par conséquent  $x'_i \in I - J_{\varepsilon i}$  pour tout  $i = 1, 2, \dots, n$  et,  $p' \in \bar{G}_\varepsilon$  entraînant d'après (8)  $x'_i \in \bar{J}_{\varepsilon i}$  pour tout  $i = 1, 2, \dots, n$ , on conclut que  $x'_i \in \text{Fr}(\bar{J}_{\varepsilon i})$  pour tous ces indices, d'où  $p' \in S_\varepsilon$ .

**COROLLAIRE.** *Les composantes de l'ensemble  $A_n$  sont, pour tout  $n \geq 2$ , des courbes régulières d'ordre  $\leq 2^n$  au sens fort.*

Cette évaluation ne se laisse pas améliorer, l'ensemble  $A_n$  pour les fonctions  $f_i(x_i) = (x_i - 2^{-1})^2$  où  $i = 1, 2, \dots, n$  étant d'ordre exactement  $2^n$  au point  $p = (2^{-1}, 2^{-1}, \dots, 2^{-1})$ . Aussi, en vertu du théorème (du corollaire), la courbe composée de  $C$  (voir p. 170) et de ses images de rotation de  $90^\circ$ ,  $180^\circ$  et  $270^\circ$  autour du point  $p$ , dans lequel elle est d'ordre 4, mais pas au sens fort, n'est-elle un ensemble  $A_2$  (pas plus qu'une composante de cet ensemble) pour aucun couple de fonction  $f_1, f_2$  satisfaisant à (2). Cependant, le problème réciproque suivant reste ouvert: existe-t-il pour tout  $A$  compact (ou étant une courbe) d'ordre  $2^n$  au sens fort un système de  $n$  fonctions continues  $f_1, f_2, \dots, f_n$  pour lesquelles  $A_n$  (ou une composante de  $A_n$ ) coïncide avec  $A$ ?

Ajoutons pour terminer que si les fonctions  $f_1, f_2, \dots, f_n$  sont en outre monotones par intervalles — bien entendu, dont la somme finie couvre  $I$  (cas particulier envisagé dans [4]), il n'y a dans  $A_n$  qu'un nombre fini des points d'ordre  $\geq 3$ .

Pour le montrer, subdivisons  $I$  en sous-intervalles par les bouts  $b_0 = 0, b_1, \dots, b_m = 1$  de ceux de monotonie de toutes les  $n$  fonctions, ayant rangé ces bouts dans leur ordre de croissance. Les intervalles  $b_k < x < b_{k+1}$  où  $k = 1, 2, \dots, m-1$  sont donc ceux de monotonie de toutes ces fonctions simultanément. Soit  $B$  l'ensemble de leurs bouts; c'est donc un ensemble fini.  $M$  désignant l'ensemble des points  $p = (a_1, a_2, \dots, a_n)$  de  $A_n$  dont toutes les coordonnées appartiennent à  $I - B$ , il est aisé de voir que les fonctions  $f_1, f_2, \dots, f_n$  étant monotones par intervalles, l'ensemble  $A_n - M$  est également fini. Or  $A_n$  est d'ordre  $\leq 2$ , et même au sens fort, en tout point  $p \in M$ .

En effet, il existe en vertu de (v) un  $\varepsilon = \min_{i=1,2,\dots,n} m_{a_i}$  positif et tel que  $J_{\varepsilon i} \subset I - B$  pour  $i = 1, 2, \dots, n$ . Toute fonction  $f_i$  étant donc monotone dans  $J_{\varepsilon i}$ , ses valeurs aux points de  $\text{Fr}(J_{\varepsilon i})$ , c'est-à-dire aux bouts de  $J_{\varepsilon i}$ , sont distinctes. En vertu de (iii), l'une d'elles est donc égale à  $f_i(a_i) - \varepsilon$  et l'autre à  $f_i(a_i) + \varepsilon$ . En désignant ces bouts de  $J_{\varepsilon i}$  par  $x_i^-$  et  $x_i^+$  respectivement, il n'y a par conséquent dans  $S_\varepsilon$ , d'après (8), que 2 points de  $A_n$ , à savoir le point  $x_1^-, x_2^-, \dots, x_n^-$  et le point  $x_1^+, x_2^+, \dots, x_n^+$ .

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# Über eine gewisse Linearisationsmethode der Differentialgleichungen vom Pendeltypus

von

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In der vorliegenden Arbeit wird gezeigt, wie man eine lineare Differentialgleichung konstruiert, die in einem gewissen Sinne am besten eine gegebene Gleichung vom Pendeltypus approximiert.

1. Wir bezeichnen mit  $y_1(t)$  und  $y_2(t)$  zwei auf dem geschlossenen Intervall  $\langle 0, T \rangle$  stetige Funktionen, die in den Punkten  $t_k$ ;  $0 = t_0 < t_1 < \dots < t_{n+1} = T$ ;  $n \geq 0$  einander gleich sind. Wir setzen voraus, dass diese Funktionen ausserhalb der Punkte  $t_k$  ( $k = 1, \dots, n$ ) der Klasse  $C_2$  angehören; in den Punkten  $t_k$  können die ersten und zweiten Ableitungen endliche Sprünge aufweisen.

Es mögen nun die Funktionen  $y_1(t)$  und  $y_2(t)$  überall, mit Ausnahme der Punkte  $t_k$ , den Differentialgleichungen genügen:

$$(1) \quad \ddot{y}_1 = f_1(y_1(t)); \quad \ddot{y}_2 = f_2(y_2(t)).$$

Ausserdem nehmen wir an, dass die zusammengesetzte Funktion  $f_2(y_1(t))$  stetig ist. Man kann somit den folgenden Satz beweisen:

SATZ 1. Wenn die Zahl  $N$

$$-N = \inf_{\substack{t \in (0, T) \\ y_1(t) \neq y_2(t)}} \frac{f_2(y_1(t)) - f_2(y_2(t))}{y_1(t) - y_2(t)}$$

der Ungleichung

$$0 \leq N < w^2; \quad w = \frac{\pi}{\max_{k/0 \dots n} |t_{k+1} - t_k|}$$

genügt, dann erfüllt die Grösse

$$\Delta_{12} = \|\dot{y}_1 - \dot{y}_2\|$$

die Ungleichung

$$(2) \quad \Delta_{12} \leq \frac{w}{w^2 - N} \|f_2(y_1) - f_1(y_1)\|.$$

Mit  $\|h(t)\|$  bezeichnen wir  $\left(\frac{1}{T} \int_0^T h^2(t) dt\right)^{1/2}$ .

2. Die Funktion  $y_0(t)$  sei die Lösung der ersten Randwertaufgabe für die Gleichung

$$(3) \quad \ddot{y} + g(y) = 0; \quad g(y) \in C(-\infty, +\infty)$$

auf dem Intervall  $\langle 0, T \rangle$ . Es sei die Funktion  $y_{ap}(t, a, b)$  die Lösung der Approximationsgleichung

$$(4) \quad \ddot{y} + ay + b = 0$$

mit den Bedingungen

$$(5) \quad y_{ap}(0, a, b) = y_0(0); \quad y_{ap}(T, a, b) = y_0(T); \quad y_{ap}(t_k, a, b) = y_0(t_k).$$

Wenn der Parameter  $a$  die Ungleichung  $0 \leq a < w^2$  erfüllt, kann man zur Fehlerabschätzung der Approximationslösung  $y_{ap}$  den Satz 1 benutzen. Wir erhalten  $N = a$  und

$$(6) \quad \|\dot{y}_0 - \dot{y}_{ap}\| \leq \frac{w}{w^2 - N} \|g(y_0) - ay_0 - b\|.$$

Die Ungleichung [1]

$$\|\dot{y}_0 - \dot{y}_{ap}\| \geq \frac{2}{T} \max_{t \in 0, T} |y_0 - y_{ap}|$$

erlaubt aus der Ungleichung (6) die Fehlerabschätzung in der Metrik  $C$  zu erhalten. Wir stellen nun die Aufgabe der „besten linearen Approximation“ der Gleichung (3):

Im Bereich  $0 < a < w^2$ ,  $-\infty < b < +\infty$  sind  $a$  und  $b$  so zu wählen, dass die rechte Seite der Abschätzung (6) am kleinsten ist.

Unter zusätzlichen Voraussetzungen zeigen wir im nächsten Paragraphen, wie diese Aufgabe zu lösen ist.

3. Wir setzen voraus, dass die Funktion  $y_0(t)$  im Intervall  $\langle 0, T \rangle$  streng monoton ist. Dann kann man im Integral

$$\|g(y_0) - ay_0 - b\|^2$$

die Integrationsvariable  $t$  durch  $x = y_0(t)$  ersetzen. Mit Hilfe des ersten Integrals der Gleichung (3)

$$\frac{1}{2} \dot{y}_0^2 + G(y_0) = E = \text{const.},$$

wobei  $G(x) = \int g(x) dx$ , erhalten wir die Identität

$$(j) \quad \|g(y_0) - ay_0 - b\|^2 = \frac{1}{T} \int_{y_0(0)}^{y_0(T)} \frac{(g(x) - ax - b)^2}{2(E - G(x))} dx.$$

Wir führen die Polynome  $\varphi_0 = 1$  und  $\varphi_1$  des Grades 0 und 1 ein, die orthogonal nach dem Gewicht  $p(x)$  sind, welches den Bedingungen

$$(7) \quad \int_{y_0(0)}^{y_0(T)} p(x) dx = 1; \quad k^2 p(x) \geq \frac{1}{T \int 2(E - G(x))}$$

genügt, wobei  $k$  eine gewisse Konstante bedeutet. Falls wir

$$ax + b = \alpha \varphi_1 + \beta \varphi_0; \quad a = \alpha \varphi_1'; \quad b = \alpha \varphi_1(0) + \beta$$

setzen, erhalten wir aus der Gleichung (j) und den Bedingungen (7) die Abschätzung

$$(k) \quad \|g(y_0) - ay_0 - b\|^2 \leq k^2 [(gg) + \alpha^2 + \beta^2 - 2\alpha g_1 - 2\beta g_0],$$

wobei die Abkürzung  $((k(x)l(x)))$  das Integral  $\int p(x)k(x)l(x)dx$  bedeutet, die Grössen  $g_1 = (g\varphi_1)$ ,  $g_0 = (g\varphi_0)$  dagegen die Koeffizienten der Funktion  $g(x)$  in der Entwicklung nach den Polynomen  $\varphi_0$  und  $\varphi_1$  darstellen.

Die Abschätzung des Approximationsfehlers (6) können wir mit Hilfe von (k) als

$$(8) \quad \|\dot{y}_0 - \dot{y}_{ap}\| \leq \frac{k w}{\varphi_1'} 0(\alpha, \beta)$$

aufschreiben, wobei

$$0(\alpha, \beta) = \frac{1}{m - \alpha} ((gg) + \alpha^2 + \beta^2 - 2\alpha g_1 - 2\beta g_0)^{1/2};$$

$$m = \frac{w^2}{\varphi_1'}.$$

LEMMA. Für die Funktion  $0(\alpha, \beta)$  gilt folgendes:

1. Wenn  $m \neq g_1$ , dann erreicht die Funktion  $0(\alpha, \beta)$  ihr Extremum im Punkte

$$(9) \quad \alpha = \alpha_0 \equiv g_1 - \frac{(gg) - g_1^2 - g_0^2}{m - g_1}; \quad \beta = \beta_0 \equiv g_0;$$

2. Die Ungleichung  $m > \alpha_0$  besteht dann und nur dann, wenn  $m > g_1$ ;

3. Falls  $m > g_1$ , erreicht die Funktion  $0(\alpha, \beta)$  im Extrempunkt für den Bereich  $0 \leq \alpha < m$ ,  $-\infty < \beta < +\infty$  ihr absolutes Minimum.

Nun lösen wir die Aufgabe der „besten Approximation“:

SATZ 2. Für  $m > g_1$  und  $\alpha_0 \geq 0$  finden wir die beste Abschätzung des Fehlers  $\|\dot{y}_0 - \dot{y}_{ap}\|$ , wenn wir  $\alpha = \alpha_0$ ,  $\beta = \beta_0$  setzen. Wir erhalten dann für die Näherungslösung  $y_{ap}(t, \alpha_0, b_0)$ , wobei  $\alpha_0 = \alpha_0 \varphi'_1$ ,  $b_0 = \alpha_0 \varphi_1(0) + \beta_0$ , die Abschätzung

$$(10) \quad \|\dot{y}_0 - \dot{y}_{ap}\| \leq \frac{kw}{\varphi'_1} \frac{d}{\sqrt{1+d^2}}; \quad d = \frac{((gg) - g_0^2 - g_1^2)^{1/2}}{m - g_1}.$$

Der Beweis folgt aus der Ungleichung (8) und dem Lemma über die Funktion  $(\alpha, \beta)$ .

Falls die Abschätzung (10) zufriedenend ist — d. h. falls die Grösse  $d$  klein ist — kann man die beste Abschätzung durch eine „praktische“ ersetzen, indem man  $\alpha = \alpha_p \equiv g_1$ ,  $\beta = \beta_p \equiv g_0$  setzt. In der Abschätzung (10) fällt dann  $\sqrt{1+d^2}$  fort.

Die Abschätzung (10) hängt von der Wahl des Gewichtes  $p(x)$  ab und ist dann am besten, wenn

$$p(x) = \frac{1}{T} \frac{1}{\sqrt{2(E - G(x))}}; \quad k = 1$$

gesetzt wird. Diese Wahl kann aber manchmal die Berechnung der Integrale  $(gg)$ ,  $g_1$  usw. erschweren. Zeitweilig kann es sich also als zweckmässig erweisen, vereinfachter Rechnungen willen, sich mit einer etwas schlechteren Abschätzung befriedigt zu erklären.

4. Zur Illustration linearisieren wir nach der angegebenen Methode die Pendelgleichung  $\ddot{y} + \sin y = 0$  unter den Bedingungen  $y(0) = -x_m$ ,  $y(T) = +x_m$ ,  $T = 2K\left(\sin \frac{x_m}{2}\right)$ . Die einzige genaue Lösung ist die Funktion  $am(t)$  [2]. Als Gewichtsfunktion nehmen wir

$$p(x) = \frac{1}{\pi} \frac{1}{\sqrt{x_m^2 - x^2}}$$

an. Für  $k^2 = \frac{\pi}{T} \sqrt{\frac{x_m}{\sin x_m}}$  sind dann die Bedingungen (7) erfüllt. Die Polynome  $\varphi_0, \varphi_1$  sind hier unnormierte Tschebyscheffschen Polynome. Falls wir die „praktische“ Approximation anwenden, erhalten wir nach einigen Rechnungen die linearisierte Gleichung

$$\ddot{y} + \frac{2J_1(x_m)}{x_m} y = 0; \quad (J_1(x_m) - \text{Besselsche Funktion})$$

deren Lösung

$$y_{ap} = x_m \frac{\sin at}{\sin aT}; \quad a^2 = \frac{2J_1(x_m)}{x_m}$$

ist [3]. Aus den angegebenen Fehlerabschätzungen erhalten wir nach Umrechnung die Ungleichung

$$\max_{t \in \langle 0, T \rangle} |am(t) - y_{av}| \leq k \frac{\frac{\pi}{2}}{\left(\frac{2\pi}{T}\right)^2 - a} \left( \frac{1}{2} - \frac{1}{2} J_0(2x_m) - 2 J_1^2(x_m) \right)^{1/2}.$$

Anschliessend sind einige numerische Ergebnisse angegeben.

$x_m$	$a$	Fehler (kleiner als)	Prozentfehler
0.6	0.955670	$10^{-2} \times 0.343$	$1\% \times 0.28$
1.0	0.880101	1.677	0.84
1.5	0.743915	6.256	2.08

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# On the Representation of Homotopic Classes by Regular Functions

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**Introduction.** In a lecture to the London Mathematical Society in 1956, Professor Kuratowski mentioned a conjecture equivalent to the following ([1], p. 319 footnote and p. 344, theorem 5):

**THEOREM.** *Let  $A$  be an open set in the complex plane,  $f(x)$  a complex valued continuous function in  $A$  which does not take the values 0 or  $\infty$ . Then there exists a function  $F(x)$ , regular in  $A$ , and a function  $g(x)$ , continuous in  $A$  such that*

$$f(x) = F(x)e^{g(x)}.$$

After some time I succeeded in finding a proof of this theorem which I communicated to Professor Kuratowski, who returned to me a version of my original proof greatly simplified by Professor Sikorski. This simplified proof is produced in this paper, and I am greatly obliged to Professors Kuratowski and Sikorski for their permission to publish it.

The above theorem is obviously equivalent to the following statement (see [2]):

*Let  $A$  be an open set in the complex plane, let  $B$  be the set of all finite complex  $x \neq 0$ . Consider the continuous functions which map  $A$  into  $B$ . Then each homotopy class of these functions contains a function regular in  $A$ .*

This statement suggests the following

**PROBLEM.** *Let  $A$  and  $B$  be any two open sets on the complex sphere. Consider all homotopy classes of continuous functions which map  $A$  into  $B$ . Is it true that each homotopy class contains a function meromorphic in  $A$ ?*

The answer is as follows.

1. *If  $A$  and  $B$  are identical with the whole complex sphere, the answer is negative.* For the only functions meromorphic on  $A$  are the rational functions. Their mapping of  $A$  into  $B$  has non-negative (Brouwer) degree and it is easy to construct continuous functions which map  $A$  into  $B$  with negative degree.

2. In all other cases, where either  $A$  or  $B$  is simply connected, there exists only one homotopy class, so the answer is trivially in the affirmative. This result can be proved in a variety of ways. As the proofs are purely topological they are omitted from this paper.

3. If  $A$  is at least doubly connected and  $B$  consists of the whole plane with two points removed, the answer is in the affirmative.

4. If  $A$  is at least doubly connected and if  $B$  is either at least trebly connected or doubly connected in such a way that  $B + b$  is also at least doubly connected for each individual point  $b$ , then the answer is negative.

There remains the case when  $B$  can be mapped conformally onto the set  $B_0: |x| < 1, x \neq 0$ . In this case the answer is negative if  $A$  is not simply connected and if there exists an open set on the sphere which intersects the complement of  $A$  in a non-void set of linear measure 0; in particular, if the complement of  $A$  contains isolated points. On the other hand, the answer is affirmative if  $A$  is of finite connectivity and if the complement of  $A$  does not contain isolated points.

But if  $A$  is of infinite connectivity, the problem has not been completely solved. Using the above theorem, we can state in the following form this unsolved

**PROBLEM.** *Let  $A$  be an open set in the complex plane of infinite connectivity. Is it always possible to find a non-vanishing function  $f(x)$ , regular in  $A$ , such that  $\log|f(x)| + h(x)$  is unbounded above in  $A$  for each real function  $h(x)$  harmonic in  $A$ ?*

**Preparations.** We require the following extension of Riemann's mapping theorem.

*Every doubly connected open set can be mapped conformally on one and only one of the following three sets:*

- a)  $x \neq 0, \infty$ ,
- b)  $|x| < 1, x \neq 0$ ,
- c)  $1 < |x| < R, R > 1$ .

**RUNGE'S THEOREM.** *Let  $A$  be an open set and let  $f(x)$  be regular on  $A$ . Then there exists a sequence of rational functions which converge uniformly to  $f(x)$  on every closed subset of  $A$ . Moreover, the poles of these rational functions may be confined to a set  $T$ , if  $T$  has at least one point in common with every topological component of the complement of  $A$  (see [3], p. 176).*

We shall apply the theorem only for sets  $A$  of finite connectivity.

**MONTÉL'S THEOREM.** *If the functions  $f_n(x)$  ( $n = 1, 2, 3, \dots$ ) are regular,  $\neq 0, \neq 1$  on an open set  $A$ , then there exists a subsequence which converges uniformly in every closed subset of  $A$  towards a regular function  $F(x)$ . Moreover,  $F(x)$  is either constant (possibly infinite) or  $F(x) \neq 0, F(x) \neq 1$  (see [3], p. 350).*

Finally we require a

**LEMMA.** *Let  $A$  be an open set in the complex plane. Then there exists a sequence of open subsets  $A_1, A_2, A_3 \dots$  such that*

- (i)  $A_{n+1}$  contains the closure of  $A_n$ .
- (ii) The union of  $A_1, A_2, A_3 \dots$  equals  $A$ .
- (iii)  $A_n$  is of finite connectivity.
- (iv) For each  $n \geq 1$  each topological component of the complement of  $A_n$  contains a point not in  $A$ .

This lemma can be easily proved by covering the plane with square gratings of decreasing mesh. The details are left to the reader.

**Proof of the Theorem.** We construct a sequence of open sets  $A_1, A_2 \dots$  as outlined in the Lemma. It is almost obvious that our theorem is true on a set  $A_n$  of finite connectivity. More precisely, we can say that for all  $x$  on  $A_n$

$$(1) \quad f(x) = r_n(x) e^{g_n(x)},$$

where the  $r_n(x)$  are rational functions with poles outside  $A$ . If we compare (1) for  $n$  and  $n+1$ , we obtain

$$e^{g_{n+1}(x) - g_n(x)} = r_n(x) r_{n+1}^{-1}(x)$$

on  $A_n$ . Hence,  $g_{n+1}(x) - g_n(x)$  is regular on  $A_n$ , and by virtue of Runge's Theorem we can find a rational function  $p_n(x)$  such that

$$|g_{n+1}(x) - g_n(x) - p_n(x)| < 2^{-n}$$

on  $A_n$ , and such that the poles of  $p_n(x)$  do not lie in  $A$ . Hence, we have on  $A_n$  for  $m > n$

$$f(x) = r_n(x) e^{g_n(x)} = r_m(x) e^{g_n(x) + \sum_{v=n}^{m-1} [g_{v+1}(x) - g_v(x) - p_v(x)] + \sum_{v=n}^{m-1} p_v(x)}.$$

This formula can be transformed into

$$f(x) = \left[ r_m(x) e^{\sum_{v=1}^{m-1} p_v(x)} \right] \cdot \left[ e^{g_n(x) - \sum_{v=1}^{n-1} p_v(x)} \right] e^{\sum_{v=n}^{m-1} [g_{v+1}(x) - g_v(x) - p_v(x)]}.$$

The third factor tends to a limit as  $m \rightarrow \infty$  uniformly on  $A_n$ . As the second factor is independent of  $m$ , the first factor tends to a limit as  $m \rightarrow \infty$  uniformly on  $A_n$  and therefore this limit is a function  $F(x)$ , regular in  $A$ . Hence, the product of the second and the third factor tends to a limit  $e^{g(x)}$  on  $A_n$  and this proves our theorem as this last limit is independent of  $n$ .

**Further proofs.** We assume that  $A$  is at least doubly connected, and  $B$  either at least trebly connected or that  $B$  can be mapped conformally on the region

$$1 < |x| < R,$$

where  $R > 1$  is a constant. In the first case, we can assume that  $B$  does not contain 0, 1, or  $\infty$  and that there exists in  $B$  a closed continuous

curve  $z(\theta)[0 \leq \theta \leq 2\pi]$  which separates any 2 of these 3 points and which satisfies the relations

$$\int_0^{2\pi} d\arg z = 2\pi, \quad \int_0^{2\pi} d\arg(z-1) = -2\pi.$$

Clearly, such a curve cannot be continuously transformed in  $B$  into an arbitrary small neighbourhood of a point.

In the second case, we can find in  $B$  a closed simple Jordan curve  $z(\theta)[0 \leq \theta \leq 2\pi]$  which separates 0 from  $\infty$  and which satisfies the relation

$$\int_0^{2\pi} d\arg z = 2\pi.$$

Again, this curve cannot be continuously transformed in  $B$  into an arbitrarily small neighbourhood of a point.

Similarly, as  $A$  is at least doubly connected, we can assume that  $A$  does not contain 0 or  $\infty$  and that there lies in  $A$  a closed rectifiable Jordan curve  $C$  which separates 0 from  $\infty$ . Introducing in  $A$  polar co-ordinates

$$x = re^{i\theta},$$

we define for integers  $n \geq 1$

$$f_n(x) = z(n\theta),$$

which are continuous functions in  $A$ . If these functions were homotopic to functions  $F_n(x)$  regular in  $A$ , we would have

$$\frac{1}{2\pi i} \int_C \frac{F'_n(x)}{F_n(x)} dx = \frac{1}{2\pi} \int_0^{2\pi} d\arg z(n\theta) = n.$$

Clearly, no subsequence of the functions  $F_n(x)$  can converge to a constant, as the curve  $C$  is mapped by them into a curve in the  $B$ -plane which cannot be contracted into a point. Therefore, by Montel's Theorem, a subsequence must converge towards a limit function  $F(x)$  regular in  $A$ , so that  $F(x)$  maps  $A$  into  $B$ . But

$$\frac{1}{2\pi i} \int_C \frac{F'}{F}(x) dx = \lim_{n \rightarrow \infty} \int_C \frac{F'_n}{F_n}(x) dx = \infty$$

which is absurd. This proves the statement 4 of the Introduction.

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## Modification of Fundamental Equation of Stellar Statistics in the Light of Discontinuity of Interstellar Matter

by

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Numerous investigators of stellar distribution use the fundamental equations of stellar statistics in a form which disregards the patchy structure of interstellar matter, taking into account only the mean value of interstellar absorption in given distance and direction. This procedure may introduce certain systematic errors the influence of which may be illustrated by the following simple example.

Let us assume that the star density in a given cone is constant and the absolute magnitude of stars contained in this cone is also constant. Owing to the discontinuity of interstellar matter the light of some stars in a given volume element would suffer greater interstellar absorption than the mean value corresponding to the given distance. In computations disregarding the patchy structure of interstellar matter, these stars will be shifted towards more distant regions. On the other hand stars the light of which suffers smaller absorption will be moved nearer to the sun. Each star from the first group will contribute less to the density of stars in distant regions than a star shifted towards the region close to the sun, because the volume of the distant region is considerably greater than the volume of the region lying close to the sun. If the frequency function of interstellar absorption (in a given distance) is symmetrical as regards its mean value, we should get, contrary to accepted assumption, an excess of star density in the neighbourhood of the sun, and a deficiency in the regions remote from it.

To avoid errors of this kind, the authors propose certain modifications of the fundamental equations of stellar statistics, taking into account the influence of discontinuity of interstellar absorption owing to the patchy structure of matter responsible for this effect.

Let us consider, as usual, stars contained in volume element  $r^2 dr$  in a cone of unit solid angle, in distance  $r$  from the top of the cone.

The number  $dN$  of stars of apparent magnitude  $m \pm \frac{1}{2}dm$ , the light of which suffers interstellar absorption amounting to  $\varepsilon(x \pm \frac{1}{2}dx)$  may be represented by the following formula:

$$dN = D(r)\varphi(M)p(x)r^2drdm dx,$$

where  $M = m - 5\log r + 5 - \varepsilon x$ ,  $D(r)$  denotes the star density,  $\varphi(M)$  luminosity function,  $p(x)$  the probability law of intersection by the line of sight  $x$  clouds of interstellar matter,  $\varepsilon$  mean absorption of a cloud. The total number of stars of apparent magnitude  $m \pm \frac{1}{2}dm$  thus amounts to:

$$A(m)dm = dm \int_0^\infty D(r)r^2dr \int_0^\infty \varphi(M)p(x)dx.$$

Let  $\varphi(M)$  develop into a series in the neighbourhood of  $x = x_0$  corresponding to the mean number of intersected clouds  $x_0$ .

$$A(m) = \int_0^\infty D(r)r^2 \left[ \varphi(M^0) + \frac{\partial \varphi(M^0)}{\partial x} \mu'_1 + \frac{1}{2!} \frac{\partial^2 \varphi(M^0)}{\partial x^2} \mu'_2 + \dots \right] dr,$$

where  $M^0 = m - 5\log r + 5 - \varepsilon x_0$ . Here  $\mu'_1, \mu'_2, \dots$  denote the central moments of  $p(x)$ . Assuming  $p(x)$  to be Poisson's law we get:

$$\mu'_1 = 0; \quad \mu'_2 = \mu'_3 = x_0.$$

Introducing the mean interstellar absorption  $a_0(r) = \varepsilon x_0$ , we obtain, after some simple transformations, the modified form of fundamental equation:

$$(1) \quad A(m) = \int_0^\infty D(r)r^2 \varphi(M^0) \left\{ 1 + \frac{a_0(r) \cdot \varepsilon}{2\varphi(M^0)} \left[ \frac{\partial^2 \varphi(M^0)}{\partial M^2} - \frac{\varepsilon}{3} \frac{\partial^3 \varphi(M^0)}{\partial M^3} + \dots \right] \right\} dr.$$

In many cases it seems reasonable to postpone terms with second and higher exponents of  $\varepsilon$  and rewrite the above equation in the more simple form:

$$(2) \quad A(m) = \int_0^\infty D(r)r^2 \varphi(M^0) \left[ 1 + \frac{a_0(r)\varepsilon}{2\varphi(M^0)} \cdot \frac{\partial^2 \varphi(M^0)}{\partial M^2} \right] dr.$$

In numerical calculations, we may get the derivatives of luminosity function replacing this later by an interpolation formula:

$$\log \varphi(M) = A + BM + CM^2.$$

Assuming further that  $\varepsilon = 0^m 27$ , in accordance with W. Ambartsumian's investigations [1], and taking as the luminosity function that of van Rhijn and Luyten [2], we compute the correction term:

$$k = \frac{\varepsilon}{2\varphi(M)} \left[ \frac{\partial^2 \varphi(M)}{\partial M^2} - \frac{\varepsilon}{3} \frac{\partial^3 \varphi(M)}{\partial M^3} \right].$$

TABLE I

<i>M</i>	-6	-5	-4	-3	-2	-1	0	+1	+2
<i>K</i>	+0.24	+0.21	+0.19	+0.17	+0.15	+0.12	+0.11	+0.09	+0.07
<i>M</i>	+3	+4	+5	+6	+7	+8	+9	+10	+11
<i>K</i>	+0.06	+0.04	+0.03	+0.02	+0.01	0.00	0.00	-0.01	-0.01
<i>M</i>	+12	+13	+14	+15	+16	+17	+18	+19	+20
<i>K</i>	-0.01	-0.01	-0.01	0.00	0.00	+0.01	+0.02	+0.03	+0.04

To show the effect of introducing a correction term, we computed by the aid of the well known Kapteyn-Bok scheme the  $A'(m)$  values without this term and  $A(m)$  according to (1) assuming  $D = \text{const.}$  and adopting the interstellar absorption:

$$a_0(r) = ar \quad a = 0^m.0025 \text{ ps}^{-1}$$

corresponding to the galactic latitude zero.

Table II gives  $\log A(m)$  and the ratio  $A(m) : A'(m)$ . In addition, we give the observed mean  $A(m)_{ob}$  for  $b = 0^0$ , according to P. Parenago's compilation [3]. It is worth while to note that the observed curve increases more rapidly than that corresponding to  $D = \text{const.}$  which may be regarded as confirmation of the well known suggestion that the sun is situated in the region of relatively low star density.

TABLE II

<i>m</i>	5 <sup>m</sup>	6 <sup>m</sup>	7 <sup>m</sup>	8 <sup>m</sup>	9 <sup>m</sup>	10 <sup>m</sup>	11 <sup>m</sup>	12 <sup>m</sup>
$\log A(m)/A'(m)$	+0.03	+0.03	+0.04	+0.05	+0.05	+0.05	+0.05	+0.06
$\log A(m)$	8.60	9.08	9.55	0.00	0.42	0.82	1.20	1.55
$\log A(m)_{ob}$	8.69	9.14	9.59	0.03	0.47	0.91	1.33	1.75
<i>m</i>	13 <sup>m</sup>	14 <sup>m</sup>	15 <sup>m</sup>	16 <sup>m</sup>	17 <sup>m</sup>	18 <sup>m</sup>	19 <sup>m</sup>	20 <sup>m</sup>
$\log A(m)/A'(m)$	+0.07	+0.07	+0.07	+0.08	+0.08	+0.09	+0.10	+0.11
$\log A(m)$	1.88	2.18	2.45	2.71	2.94	3.16	3.36	3.55
$\log A(m)_{ob}$	2.16	2.55	2.91	3.26	3.58	3.88	4.16	4.41

In some cases, another procedure which seems to be more effective and economic may be recommended — namely, when we have at our disposal former analyses of star counts completely disregarding interstellar absorption, or disregarding its discontinuity.

We divide the cone into shells with radii  $r_k$  satisfying the relation:

$$\log r_k = 0.2k\varepsilon,$$

where  $\varepsilon$  is, as previously, the mean absorption in a cloud of interstellar mater,  $k = \pm 0, 1, 2, 3 \dots$ . Let  $D_k^*$  be the uncorrected star density disregarding the interstellar absorption,  $D_k$  the corrected density. Among stars contained in  $k$  shell,  $V_k D_k p_k$  (1) would, in a computation disregarding interstellar absorption, find themselves in  $k+1$  shell, because their light intersects one cloud;  $V_k D_k p_k$  (2) in  $k+2$  shell because their light intersects two clouds etc.  $p_k(j)$  is the probability of intersection of  $j$  clouds by a star in  $k$  shell.

Thus the uncorrected density may be connected with the corrected density by means of a formula:

$$(3) \quad D_k^* V_k = \sum_{j=0}^{\infty} D_{k-j} V_{k-j} p_{k-j}(j),$$

where

$$V_k = C \cdot 10^{0.6k\varepsilon} \quad C = \text{const.}$$

From a system of linear equations (3) we may easily find the unknowns  $D_1, D_2, \dots$ , the sooner because each successive equation contains only one new unknown. The probabilities  $p_{k-j}(j)$  may be identified with Poisson's law,

$$p_{j,k-j}(j) = e^{-\lambda_{k-j}} \frac{\lambda_{k-j}^j}{j!},$$

numerical values of which may be found in statistical tables.  $\lambda$  denotes the mean number of intersected clouds by the light of a star in the corresponding shell.

If the results of former analysis take into account the mean interstellar absorption in any way, we may compute the  $D^*$  values corresponding to the case disregarding absorption, transforming the distance argument in the usual manner [4], and later find the corrected densities by means of the Eq. (3).

Finally, it is worth while to note that certain modifications, due to the patchy structure of interstellar matter, are to be introduced also in the well known Oort-Vashakidse method and into its modification introduced by B. Bok and D. MacRea [5]. Starting in this method from Eq. (2) (instead of the form previously used) we obtain the following formulae fundamental for this method:

$$A_b(m_{90}) = \frac{1}{\sin^3 b} \left[ A_{90}(m) + \frac{\varepsilon a_{90}}{2} \left( \frac{a_b}{a_{90}} - 1 \right) \frac{\partial^2 A_{90}(m)}{\partial m^2} \right]$$

$$m_{90} = m + \log \sin b - (a_b - a_{90}),$$

where  $a_{90}$  and  $a_b$  denote the mean total interstellar absorption in the direction to the galactic pole and in a given direction of galactic latitude,  $b$ .  $\varepsilon$  is the mean absorption in a cloud of interstellar matter. In some cases, when  $a_b$  is considerably greater than  $a_{90}$ , results obtained by these formulae may differ from those obtained by the previous ones.

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# Čerenkov Radiation of Polarized Electrons

by

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## Introduction

Recent discoveries of non-conservation of parity in many weak interactions and theoretical suggestions of a "longitudinal" neutrino, turned the attention of many physicists to the problem of the electron polarization [2], [4]-[9]. Of particular interest are the investigations of the electron polarization in  $\beta$ -decays,  $\mu$ -meson and  $K$ -meson decays. Experimental physicists are very much interested in several electromagnetic processes with polarized electrons. The aim of this paper is to answer the question, whether it is possible to determine the electron polarization by measuring the polarization of Čerenkov radiation.

## Formulation of the problem

The state of a free electron of momentum  $\vec{p}$  is determined by the function  $\mathcal{P}$  which is an eigenfunction of the following projection operator for the electron polarization  $\mathcal{A}$ :

$$(1) \quad \mathcal{A} = \frac{1}{2}[1 + \xi \vec{\sigma} + \beta \vec{\eta} \vec{\sigma}],$$

where  $\vec{\xi}$  denotes the longitudinal and  $\vec{\eta}$  the transverse components of the unit vector of electron polarization

$$\vec{\eta} \vec{p} = 0; \quad \xi^2 + \eta^2 = 1.$$

The polarization of the photon is described by the vector

$$(2) \quad \vec{e} = a \vec{e}_1 + e^{i\delta} b \vec{e}_2; \quad a^2 + b^2 = 1.$$

Here  $\vec{e}_1$  and  $\vec{e}_2$  are two mutually perpendicular unit vectors which are also perpendicular to the momentum vector of the photon. The parameters  $a$  and  $\delta$  determine the type of the photon polarization.

Let us now consider the Čerenkov effect. The probability of the transition of an electron from a state of momentum  $\vec{p}$  and polarization given by  $\vec{\xi}$  and  $\vec{\eta}$  to a state of momentum  $\vec{p}'$  and polarization given by  $\vec{\xi}'$  and  $\vec{\eta}'$  with a simultaneous emission of a photon of momentum  $\vec{k}$  and polarization given by  $a$  and  $\delta$  is a function of all these parameters

$$(3) \quad W = W(\vec{p}, \vec{p}', \kappa; \vec{\xi}, \vec{\eta}, \vec{\xi}', \vec{\eta}', a, \delta),$$

where  $\kappa$  is the refractive index of the medium.

For fixed initial and final momenta the polarization in the final state is described by four parameters; two  $(a, \delta)$  related to the photon and two  $(\eta'_1, \eta'_2)$  related to the electron. Thus, we see that for fixed momenta the function  $W$  depends on six independent polarization parameters

$$(4) \quad W = W(\eta_1, \eta_2; \eta'_1, \eta'_2, a, \delta),$$

where  $\eta_1$  and  $\eta_2$  determine the polarization of the electron in the initial state. The method of polarization calculations adopted in [1] can now be applied, making it possible to find out the actual polarizations in the final state from the condition for an extremum of  $W$ . The values of  $\eta'_1, \eta'_2, a$  and  $\delta$  for which the function  $W$  has a maximum will, of course, depend on the initial polarization, i. e. on  $\eta_1$  and  $\eta_2$ . Thus, we have the following set of equations which determine the extremal values of  $\eta'_1, \eta'_2, a, \delta$ :

$$(5) \quad \begin{aligned} \frac{\partial W}{\partial \eta'_1} \Big|_{\eta'_2, a, \delta} &= 0; & \frac{\partial W}{\partial \eta'_2} \Big|_{\eta'_1, a, \delta} &= 0, \\ \frac{\partial W}{\partial a} \Big|_{\eta'_1, \eta'_2, \delta} &= 0; & \frac{\partial W}{\partial \delta} \Big|_{\eta'_1, \eta'_2, a} &= 0. \end{aligned}$$

If it turns out that it is possible to measure  $a$  and  $\delta$  then it will be also possible to compare experimental values of  $a$  and  $\delta$  with the calculated dependence  $a = a(\eta_1, \eta_2)$  and  $\delta = \delta(\eta_1, \eta_2)$ . Thus, in principle it is possible to find the polarization of the electrons by measuring photon polarization in Čerenkov effect.

#### Calculation of the function $W$

Let us consider Čerenkov effect in the first order (with respect to  $e$ ) approximation of the transition matrix  $\hat{S}$ . In this approximation, the probability of the transition is proportional to

$$(6) \quad |\psi_f^+(\vec{p}')(\hat{a}\vec{\epsilon})\psi_i(\vec{p})|^2,$$

$\psi_i(\vec{p})$  is an eigenfunction of the projection operator (1);  $\psi_f(\vec{p}')$  is an eigenfunction of a similar projection operator but with  $\vec{\xi}$  and  $\vec{\eta}$  replaced by

$\vec{\xi}'$  and  $\vec{\eta}'$ . By means of the projection operators (1) calculation of (6) can be now reduced to the standard calculation of the spurs of matrices:

$$(7) \quad |\psi_f^+(\vec{p}')(\hat{\alpha}\vec{e})\psi_i(\vec{p})|^2 = \frac{1}{32 E(\vec{p}) E(\vec{p}')} \cdot \text{Sp} \{ (1 + \vec{\xi}\vec{\sigma} + \beta\vec{\eta}\vec{\sigma}) \times \\ \times [(\hat{\alpha}\vec{p}) + \beta m + E(\vec{p})](\hat{\alpha}\vec{e}^*) (1 + \vec{\xi}'\vec{\sigma} + \beta\vec{\eta}'\vec{\sigma}) \times \\ \times [(\hat{\alpha}\vec{p}') + \beta m + E(\vec{p}')](\hat{\alpha}\vec{e}) \} \\ \vec{p}' = \vec{p} - \vec{k} \quad E(\vec{p}') = E(\vec{p}) - \frac{k}{\varkappa}.$$

We calculate now the spurs and expand the result in powers of  $k$  introducing the following frame of reference

$$(8) \quad \begin{aligned} \vec{p} &= [p, 0, 0], & \vec{\eta} &= [0, \eta_y, \eta_z], \\ \vec{k} &= k[\cos \theta, \sin \theta, 0], & \vec{\eta}' &= \left[ \eta'_y \frac{k}{p} \sin \theta, \eta'_y, \eta'_z \right], \\ \vec{e}_1 &= [\sin \theta, -\cos \theta, 0], & \xi' &= \pm (1 - \eta'^2)^{1/2}, \\ \vec{e}_2 &= [0, 0, 1], \end{aligned}$$

where  $\theta$  denotes the angle between  $\vec{k}$  and  $\vec{p}$ . We obtain the following expression for  $W$  as a function of  $\eta_y, \eta_z, \eta'_y, \eta'_z, a$  and  $\delta$ .

$$(9) \quad W \sim \frac{1}{4 E^2(p)} \cdot \left\{ a^2 p^2 \sin^2 \theta (1 \pm \xi \sqrt{1 - \eta_y'^2 - \eta_z'^2} + \eta_y \eta'_y + \right. \\ \left. + \eta_z \eta'_z) \left( 1 + \frac{k}{E \varkappa} \right) + a (1 - a^2)^{1/2} \sin \delta \times \right. \\ \left. \times k \left[ \left( \frac{E}{\varkappa} \cos \theta - p \right) (\xi \pm \sqrt{1 - \eta_y'^2 - \eta_z'^2}) + \frac{m}{\varkappa} \sin \theta (\eta_y + \eta'_y) \right] + \right. \\ \left. + m \frac{k}{\varkappa} \sin \theta [a (1 - a^2)^{1/2} \cos \delta (\xi \eta'_z \pm \eta_z \sqrt{1 - \eta_y'^2 - \eta_z'^2}) - \right. \\ \left. - a^2 \cos \theta (\xi \eta'_y \pm \eta_y \sqrt{1 - \eta_y'^2 - \eta_z'^2}) \right] + \\ \left. + a (1 - a^2)^{1/2} \sin^2 \theta k p \cos \delta [\eta_y \eta'_z - \eta_z \eta'_y] \right\}.$$

### Discussion

The various terms in the expression (9) are proportional to:

- I  $p^2 \sin^2 \theta,$
- II  $k \left( \frac{E}{\varkappa} \cos \theta - p \right),$
- III  $p \cdot k \sin \theta,$
- IV  $p \cdot k \sin^2 \theta.$

In the region of angles  $\theta$  far from 0 the first term I is much larger than the remaining ones. However, if terms II — IV are neglected, linear polarization of the photons ( $a = 1, \delta = 0$ ) is obtained. Hence, for angles  $\theta$  far from 0 the polarization of the photons ( $a, \delta$ ) is practically independent of the polarization ( $\eta_y, \eta_z$ ) of electrons in the initial state. Thus, the effect we are interested in (i. e. a tangible dependence of  $a$  and  $\delta$  on  $\eta_y, \eta_z$ ) should be expected within the range of very small angles  $\theta \sim 0$ . E. g. let us assume  $\sin^2 \theta \sim k/p$  and let us calculate the magnitudes of the ratios of the terms I-IV. Since

$$\cos \theta = \frac{E}{p\kappa} + \frac{k}{2p} \left(1 - \frac{1}{\kappa^2}\right),$$

we have

$$\frac{[\text{II}]}{[\text{I}]} \sim \frac{k}{p} \left[ \frac{\frac{E^2}{\kappa^2} - p^2 + \frac{Ek}{2\kappa} \left(1 - \frac{1}{\kappa^2}\right)}{p^2 - \frac{E^2}{\kappa^2} + \frac{Ek}{\kappa} \left(1 - \frac{1}{\kappa^2}\right)} \right] \sim \frac{k}{p},$$

$$\frac{[\text{III}]}{[\text{I}]} \sim \frac{k}{p} \sin \theta \sim \frac{k}{p} \sqrt{\frac{k}{p}} \ll \frac{[\text{II}]}{[\text{I}]}; \quad \frac{[\text{IV}]}{[\text{I}]} \sim \frac{k}{p} \sin^2 \theta \sim \frac{k^2}{p^2}.$$

It can easily be seen that even for small angles  $\left(\sin \theta \sim \sqrt{\frac{k}{p}}\right)$  the ratio of the largest among the terms II, III and IV to the leading term I (i. e. to 0 approximation) is of the order  $\frac{k}{p} \sim 10^{-5}$ . Thus, we see that even for very small angles the influence of the electron polarization upon Čerenkov radiation is too small to be measured with sufficient accuracy.

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## Three-Centre Integrals in Iron

by

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*Presented by L. INFELD on January 10, 1958*

In the application of the tight binding method to metals, the most difficult problem is the determination of constants in the matrix components of energy. Slater and Koster [1] suggested the use of these constants as disposable constants to be determined so as to fit the results of more accurate calculation of energy eigenvalues made by other methods. But they developed also in a systematic way the two-centre approximation for calculating these constants directly from atomic or orthogonalized orbitals, along the lines previously initiated by Fletcher and Wohlfarth [2]-[4]. The two-centre approximation has the advantage of being manageable in actual numerical calculations, but it constitutes, in fact, an unwarranted simplification in the solid state problems. First, there are usually considerably more energy  $E$ -constants than integrals of the two-centre type. In Table I [1] reducing the  $E$ -constants to the two-centre integrals is of a general character, it applies, in fact, to every crystal lattice by putting in appropriate direction cosines of the line joining two neighbours.

D. M. Edwards suggested that three-centre integrals should be taken into account in the nearest neighbour approximation. In this approach, the given specific crystal structure has a more pronounced bearing on the problem. In the two-centre approximation we look at the neighbours of a given atom; in the two-centre integrations we can take advantage of the axial symmetry about the line joining the two centres as in a diatomic molecule. Now, considering the three-centre integrals in the nearest neighbour approximation, we look at the neighbours of a given line joining two atoms. We have two atoms with an appropriate wave function centered on each, and a third centre, the ion, on which a spherically symmetric potential is centered. Thus, there is no axial symmetry about the line joining any two centres. The question now arises, how many three-centre integrals must be considered when we

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Note: The author wrote this paper during his studies in England.

limit ourselves to the orbitals of the given symmetry and to a given order of neighbours.

In view of the importance of  $3d$  electrons in the transition metals, we will think of  $3d$ -like wave functions, possibly hybridized with  $4s$  wave functions.

The hybridization of  $3d$  with  $4s$  and also with  $4p$  wave functions is suggested by some authors, e. g. by Mott and Stevens [5]. It is clear, of course, that the use of the tight binding approximation to describe it may be a very poor method indeed; in connection with this problem, see, for instance, Jones and Mott [6], and Matyas [7].

We should like to remind here that Callaway [8] applied the Slater and Koster method for interpolating his results of calculation of  $3d$  and  $4s$ -like electron levels in iron and actually, in the two-centre approximation scheme.

Of transition metals iron was chosen as it has, at ordinary temperatures, the body-centred structure, which is not a close-packed cubic structure. There are fewer nearest neighbours to the given atom; thus, application of the tight binding method seems more reasonable.

In Table I, we quote the data for the first neighbours in the body-centred cubic structure. We use as the unit of length  $a/2$ , the half length of the side of the fundamental cube.

TABLE I  
Neighbours in the body-centred cubic structure

Order of neighbour	Typical position in units $a/2$	Distance in units $a/2$	Number of neighbours
1	111	$\sqrt{3}$	8
2	002	2	6

We now look at the matrix components of energy, which should be relevant for  $3d$  electrons in iron, limiting ourselves to the nearest neighbour interactions. We excise from Table II of Slater and Koster [1] the following Table II which is appropriate for body-centred cubic structure and for  $d$  and  $s$  orbitals. We omit here altogether the  $p$  wave functions. The notation of [1] is followed.

We now consider the nearest neighbours to the line (000)-(111) joining the central atom with its nearest neighbours. There are six of them in position  $R_n$ : (200), (020), (002),  $(-111)$ ,  $(1-11)$ ,  $(11-1)$ , (see Fig. 1). We centre one wave function on the central atom, another wave function on the (111) atom, the spherically symmetrical potentials  $U_n$  ( $n = 1, 2, 3, 4, 5, 6$ ) in the six positions just enumerated, and we disregard all the other lattice sites. We also omit at this point, the writing of the two-centre integrals previously considered [2], [3], [9], [10].

TABLE II

Matrix components of energy in the body-centred cubic structure for  $s$  and  $d$  orbitals

$(s s)$	$E_{s,s}(000) + 8E_{s,s}(111)\cos\xi\cos\eta\cos\zeta$
$(s xy)$	$-8E_{s,xy}(111)\sin\xi\sin\eta\cos\zeta$
$(s x^2-y^2)$	0
$(s 3z^2-r^2)$	0
$(xy xy)$	$E_{xy,xy}(000) + 8E_{xy,xy}(111)\cos\xi\cos\eta\cos\zeta$
$(xy xz)$	$-8E_{xy,xz}(111)\cos\xi\sin\eta\sin\zeta$
$(xy 3z^2-r^2)$	$-8E_{xy,3z^2-r^2}(111)\sin\xi\sin\eta\cos\zeta$
$(xz x^2-y^2)$	$4\sqrt{3}E_{xy,3z^2-r^2}(111)\sin\xi\cos\eta\sin\zeta$
$(xz 3z^2-r^2)$	$4E_{xy,3z^2-r^2}(111)\sin\xi\cos\eta\sin\zeta$
$(x^2-y^2 x^2-y^2)$	$E_{3z^2-r^2,3z^2-r^2}(000) + 8E_{3z^2-r^2,3z^2-r^2}(111)\cos\xi\cos\eta\cos\zeta$
$(3z^2-r^2 3z^2-r^2)$	$E_{3z^2-r^2,3z^2-r^2}(000) + 8E_{3z^2-r^2,3z^2-r^2}(111)\cos\xi\cos\eta\cos\zeta$
$(x^2-y^2 3z^2-r^2)$	0

The distance from the central atom is denoted by  $r = (x^2 + y^2 + z^2)^{1/2}$ , and the distance from the neighbour atom by  $r_1 = [(x-1)^2 + (y-1)^2 + (z-1)^2]^{1/2}$ . We denote by  $h(r)$  the radial part of  $s$  wave function; by  $f(r)$  that of  $d$  wave function. Considering the symmetry operations allowable in cubic structure, we can see that, among our three-centre integrals, some are equal to others. We can write the  $E(111)$  integrals in the form

$$4\pi E_{s,s} = 6 \int h(r)h(r_1) U_1 d_3x$$

$$\frac{4\pi}{15} E_{s,xy} = \int h(r)(x-1)(y-1)f(r_1)(2U_1 + U_3 + 2U_4 + U_6) d_3x$$

$$\frac{4\pi}{15} E_{xy,xy} = 2 \int xy(x-1)(y-1)f(r)f(r_1)(2U_1 + U_3) d_3x$$

$$\frac{4\pi}{15} E_{xy,xz} = 2 \int xy(x-1)(z-1)f(r)f(r_1)(U_1 + U_2 + U_3) d_3x$$

$$\frac{8\pi}{15} E_{xy,3z^2-r^2} =$$

$$= \int xy[2(z-1)^2 - (x-1)^2 - (y-1)^2]f(r)f(r_1)(2U_1 + U_3 + 2U_4 + U_6) d_3x$$

$$\frac{16\pi}{5} E_{3z^2-r^2,3z^2-r^2} =$$

$$= 2 \int (2z^2 - x^2 - y^2)[2(z-1)^2 - (x-1)^2 - (y-1)^2]f(r)f(r_1)(2U_1 + U_3) d_3x$$

It is obvious that, in order to determine the six  $E$ -constants of Table II, we have to deal, in the nearest neighbour approximation, with 16 three-centre integrals. Further reductions are possible by transforming

the spherical harmonics in the integrals to the set of axes appropriate to the nearest neighbours concerned. These reductions will be given separately.

Let us observe that, were we to take into account the  $p$  wave functions in nearest-neighbour approximation, we should have to include six more  $E$ -constants.

If we omit  $p$  functions, and go to the next-nearest neighbours of the central atom in body-centred structure, we have to deal with eleven

$E$ -constants in addition to those considered here, but the position of nearest neighbours to the line (000)—(002) are particularly symmetrical, namely (111), (1-11), (-111), (-1-11).

Finally, let us remark that the case of the hexagonal close-packed structure is particularly unfavourable to reduction, because the neighbours of the line joining the central atom with its nearest neighbours are arranged in such positions that only few symmetry operations are possible.

The author is greatly indebted to Professors N. F. Mott and H. Jones for discussions on the use of the tight binding approximation. He also thanks Dr. D. M. Edwards for suggesting the inclusion of the three-centre integrals, and Dr. E. P. Wohlfarth for several discussions.

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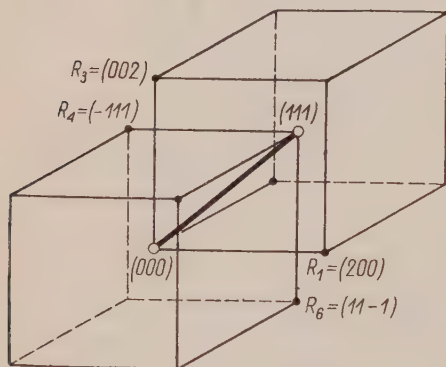


Fig. 1

## The $(p, \alpha)$ Reaction on Deformed Nuclei

by

D. CHLEBOWSKA

*Presented by L. INFELD on January 10, 1958*

In most cases that have come under observation, it has thus far not been possible to assign a satisfactory mechanism to the  $(p, \alpha)$  reaction. Experimental results point to the presence of direct interaction. Here, the latter possibility will be considered, admitting large deformation of the nuclei [1] participating in the reaction. Nonspherocity of the nuclei may not be disregarded, e. g. in considering the reaction [2]  $F^{19}(\alpha, p)Ne^{22}$ , since the  $F^{19}$  and  $Ne^{22}$  nuclei are known to have rather large coefficients of deformation, amounting to  $\varepsilon = 0.37$  and  $0.47$ , respectively [3].

The target nucleus is assumed to consist of a core and an alpha particle participating in the rotatory motion of the nucleus as a whole. The incident proton knocks the alpha particle out of the target and itself is captured onto a Nilsson orbit [4].

### Calculation of cross-section

The calculation of the cross-section is based on Born's approximation. The Coulomb interaction, that of the incident proton and the core, and also that of the emitted alpha particle and the residual nucleus are neglected. The interaction causing the reaction is given by the average potential existing between the proton and the alpha particle, which is assumed to be of the form  $V_{pa} = C\delta(\vec{r}_p - \vec{r}_a)$ , wherein  $C$  is constant. The numerical computations, with the above-mentioned effects taken into consideration, will be carried out on obtaining exact experimental data.

With the above assumptions, the matrix element for the  $(p, \alpha)$  reaction, in the center-of-mass system, is given by [5]

$$(1) \quad \begin{aligned} (f|V_{\alpha p}|i) \propto & \int d\vec{r}_{ra} d\vec{r}_{ap} d\sigma_p d\xi d\theta_i \psi_f^*(\xi, \vec{r}_{ra} + \vec{r}_{ap}) e^{-i\frac{1}{A-3}\vec{k}_a[(A-4)\vec{r}_{ra} - \vec{r}_{ap}]} \\ & \cdot V_{\alpha p}(\vec{r}_{\alpha p}) f_{m_p}''(\sigma_p) e^{i\vec{k}_0\left(\vec{r}_{\alpha p} + \frac{A-4}{A}\vec{r}_{ra}\right)} \Phi_i(\xi, \vec{r}_{ra}), \end{aligned}$$

wherein  $\xi$  denotes the internal co-ordinates of the core,  $\theta_i$  — the Eulerian angles, for

$$V_{ap}(\vec{r}_{ap}) = C\delta(\vec{r}_{ap})$$

$$(2) \quad (f|V_{ap}|i) \propto \int d\vec{r} d\sigma_p d\xi d\theta_i \psi_f^*(\xi, \vec{r}) e^{-i\frac{A-4}{A-3}\vec{k}_a\vec{r}} f_{m_p}''(\sigma_p) \cdot e^{i\frac{A-4}{A}\vec{k}_0\vec{r}} \Phi_i(\xi, \vec{r}).$$

According to A. Bohr's collective model and Nilsson's independent particle model

$$\psi_f(\xi, \vec{r}) \propto \sqrt{2I_f + 1} X_{K_f} D_{M_f K_f}^{I_f}(\theta_i)$$

with

$$(3) \quad X_{K_f} = X_{K_f}(\xi) X_{\Omega_p}(\vec{r}'),$$

$$(4) \quad X_{\Omega_p}(\vec{r}') = \sum_{l_p} \sum_{\Lambda_p + \Sigma_p = \Omega_p} R_{n_p l_p}(r) a_{l_p \Lambda_p} Y_{l_p \Lambda_p}(\vartheta', \varphi') f_{\Sigma_p}(\sigma_p)$$

and

$$\Phi_i(\xi, \vec{r}') \propto \sqrt{2I_i + 1} X_{K_i} D_{M_i K_i}^{I_i}(\theta_i),$$

where

$$X_{K_i} = X_{K_i}(\xi) g_{l_a \Omega_a}(r) Y_{l_a \Omega_a}(\vartheta', \varphi').$$

(Here, it is assumed that  $l_a$  is a good quantum number \*)).

$$(f|V_{ap}|i) \propto \sqrt{(2I_f + 1)(2I_i + 1)} \sum_{l_p} \sum_{\Lambda_p + \Sigma_p = \Omega_p} \int d\vec{r} d\sigma_p d\theta_i R_{n_p l_p}^* g_{l_a \Omega_a} \cdot \\ \cdot a_{l_p \Lambda_p}^* Y_{l_p \Lambda_p}^* f_{\Sigma_p}^* Y_{l_a \Omega_a} D_{M_f K_f}^{I_f*} D_{M_i K_i}^{I_i} f_{m_p}'' e^{i\vec{k} \vec{r}},$$

where

$$\vec{k} = \frac{A-4}{A} \vec{k}_0 - \frac{A-4}{A-3} \vec{k}_a.$$

Calculations yield

$$(f|V_{ap}|i) \propto \sqrt{(2I_f + 1)(2I_i + 1)} \sum_{l_p} \sum_{\Lambda_p + \Sigma_p = \Omega_p} \sum_{l_A} i^{l_A} \sqrt{2l_A + 1} a_{l_p \Lambda_p}^* \cdot \\ \cdot \int r^2 dr R_{n_p l_p}^*(r) g_{l_a \Omega_a}(r) j_l(kr) \int \sin \vartheta' d\vartheta' d\varphi' Y_{l_p \Lambda_p}^*(\vartheta', \varphi') Y_{l_a \Omega_a}(\vartheta', \varphi') \cdot \\ \cdot Y_{l_A}(\vartheta', \varphi') \int d\theta_i D_{M_f K_f}^{I_f*}(\theta_i) D_{M_i K_i}^{I_i}(\theta_i) D_{0 \Lambda}^l(\theta_i) D_{m_p \Sigma}^{\dagger}(\theta_i).$$

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\*) The correct function (within the framework of the independent particle model) describing the alpha particle in a deformed nucleus would be of the form

$$X_{\Omega_a} = \sum_{l_a} b_{l_a \Omega_a} g_{l_a \Omega_a}(r) Y_{l_a \Omega_a}(\vartheta', \varphi').$$

However, as the coefficients  $b_{l_a \Omega_a}$  are not known, the deformation of the initial nucleus is accounted for solely by the rotatory motion.

The radial integrals are approximated by [6]

$$\int r^2 dr R_{n_p l_p}^*(r) g_{l_a \Omega_a}(r) j_l(kr) \approx j_l(kr_0) \cdot \text{const.}$$

Ultimately, on summation over the final spins and averaging over the initial spins, the following expression for the cross-section of the  $(p, \alpha)$  reaction on deformed nuclei is obtained:

$$(5) \quad \sigma d\Omega \propto \sum_l \sum_j \left| \sum_{l_p A_p} \sum_A i^l \sqrt{\frac{(2l_a+1)2l+1}{(2l_p+1)}} \cdot j_l(kr_0) a_{l_p A_p} (l_a A \Omega_a | l_a l_p A_p) (l_a 0 0 | l_a l_p 0) \cdot \right. \\ \cdot (l \frac{1}{2} j A + \Sigma_p | l \frac{1}{2} A \Sigma_p) \cdot \\ \cdot (j I_i \frac{1}{2} + \Sigma_p K_i | j I_i I_f K_f) \left. \right|^2.$$

I am very pleased to thank Dr J. Sawicki for helpful discussions.

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# On the Derivation of Hydrodynamical Equations for the Classical Systems of Diatomic Molecules

by

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The purpose of this paper is to demonstrate the derivation of hydrodynamical equations for the diatomic fluids by the application and generalization of methods given for the classical systems of one-atomic molecules by N. N. Bogolyubov [1], [2].

1. For computations of hydrodynamical variables the distribution functions of one, two, ...  $s$  molecules were used. These functions, in an asymptotical case (i. e. when  $\Omega_v$  — the volume in the configurational space in which our system is present and  $N$  — type number of diatomic molecules of our system tends to infinity, such that  $\frac{\Omega_v}{N} = \omega = \text{const.}$ ), are defined in the following manner and have the following properties:

a) Definition

$$(1.1) \quad F_s(t, 1, 2, \dots s) = \\ = \Omega_v^s \int D(t, 1, 2, \dots N) d_{(3)} r^{(s+1,1)} d_{(3)} r^{(s+1,2)} d_{(3)} p^{(s+1,1)} d_{(3)} p^{(s+1,2)} \dots d_{(3)} p^{(N,2)},$$

where  $r_k^{(i,1)}$  and  $r_k^{(i,2)}$  describe the positions of the first and second atom of the  $i$ -th molecule, respectively, and  $p_k^{(i,1)}$  and  $p_k^{(i,2)}$  — the momenta of the same atoms.  $D$  is a distribution function of the whole system of  $N$  molecules, defined by its equation of motion

$$(1.2) \quad \frac{\partial D}{\partial t} = \\ = \sum_{i=1}^N \left\{ \frac{\partial H}{\partial r_k^{(i,1)}} \frac{\partial D}{\partial p_k^{(i,1)}} + \frac{\partial H}{\partial r_k^{(i,2)}} \frac{\partial D}{\partial p_k^{(i,2)}} - \frac{\partial H}{\partial p_k^{(i,1)}} \frac{\partial D}{\partial r_k^{(i,1)}} - \frac{\partial H}{\partial p_k^{(i,2)}} \frac{\partial D}{\partial r_k^{(i,2)}} \right\} \equiv [H; D],$$

normalization condition

$$(1.3) \quad \bar{D} = 1$$

(—denotes the integration with respect to all arguments of  $D$  from  $-\infty$  to  $+\infty$ ) and the properties of symmetry. It is required that  $P_{ij}D = D$  (where  $P_{ij}$  is the permutation operator which interchanges the  $i$ -th and  $j$ -th molecule).

b) Equation of motion. If the Hamiltonian of our systems is postulated (in an asymptotical case) in the following form

$$(1.4) \quad H = \sum_{i=1}^{1N} \left\{ \frac{p_k^{(i,1)} p_k^{(i,1)}}{2m} + \frac{p_k^{(i,2)} p_k^{(i,2)}}{2m} \right\} + \sum_{i \leq j}^N \Phi_{ij}(r_k^{(i,1)}, r_k^{(i,2)}, r_k^{(j,1)}, r_k^{(j,2)}),$$

where  $2m$  denotes the mass of the diatomic molecule, then the equations which the distribution functions must fulfill are

$$(1.5) \quad \frac{\partial F_s}{\partial t} = [H_s; F_s] + \\ + \frac{1}{\omega} \int \left[ \sum_{i=1}^s \Phi_{i,s+1}; F_{s+1} \right] d_{(3)} r^{(st1,1)} d_{(3)} r^{(st1,2)} d_{(3)} p^{(st1,1)} d_{(3)} p^{(st1,2)},$$

where  $H_s$  denotes the Hamiltonian of a cluster of  $s$  molecules.

c) The computation of average values of dynamical variables which may be presented in the following form:

$$(1.6) \quad Z = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_s \leq N} A(i_1, i_2, \dots, i_s).$$

We use the following formula

$$(1.7) \quad \bar{Z} = \frac{1}{s! \omega^s} \overline{A F_s}.$$

d) Other properties of distribution functions.

(i) They are normalized in the following way:

$$(1.8) \quad F_s = \lim_{\Omega_q \rightarrow \infty} \frac{1}{\Omega_q} \int F_{st1} d_{(3)} r^{(s+1,1)} d_{(3)} r^{(s+1,2)} d_{(3)} p^{(s+1,1)} d_{(3)} p^{(s+1,2)}, \\ \lim_{\Omega_q \rightarrow \infty} \frac{1}{\Omega_q} \int F_1 d_{(3)} r^{(1,1)} d_{(3)} r^{(1,2)} d_{(3)} p^{(1,1)} d_{(3)} p^{(1,2)} = 1,$$

(ii) In respect of the assumed property of distribution function  $D$  they fulfill the relation

$$(1.9) \quad P_{ij} F_s = F_s,$$

(iii) They must fulfill the condition of decrease of correlation, for example, in the following form

$$(1.10) \quad S_\tau^{(s)} \left\{ F_s - \prod_{i=1}^s F_1(t, r_k^{(i,1)}, r_k^{(i,2)}, p_k^{(i,1)}, p_k^{(i,2)}) \right\} \rightarrow 0, \quad \tau \rightarrow +\infty,$$

where  $S_\tau^{(s)}$  denotes the operator, which changes the arguments  $r_k^{(i,1)}, r_k^{(i,2)}$  into  $r_k^{(i,1)} - \frac{p_k^{(i,1)}}{m} \tau, r_k^{(i,2)} - \frac{p_k^{(i,2)}}{m} \tau$  respectively.

Performing the transformation

$$\begin{aligned} R_k^{(i)} &= \frac{1}{2}(r_k^{(i,1)} + r_k^{(i,2)}), & P_k^{(i)} &= \frac{1}{2}(p_k^{(i,1)} + p_k^{(i,2)}), \\ r_k^{(i)} &= r_k^{(i,1)} - r_k^{(i,2)}, & p_k^{(i)} &= p_k^{(i,1)} - p_k^{(i,2)}, \end{aligned} \quad (i = 1, 2, \dots, N)$$

we can write Eqs. (1,5) in the following form:

$$(1.11) \quad \frac{\partial F_s}{\partial t} = [H_s; F_s] + \frac{1}{\omega} \int \left[ \sum_{i=1}^s \Phi_{i,s+1}; F_{st1} \right] d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} d_{(3)} P^{(s+1)} d_{(3)} p^{(s+1)},$$

with

$$(1.12) \quad H_s = \sum_{i=1}^s \left\{ \frac{P_k^{(i)} P_k^{(i)}}{4m} + \frac{p_k^{(i)} p_k^{(i)}}{m} \right\} + \sum_{i \leq j}^s \Phi_{ij}(R_k^{(i)}, r_k^{(i)}, R_k^{(j)}, r_k^{(j)}).$$

2. The hydrodynamical quantities such as  $g$  — the density,  $gu_k$  — the momentum,  $gl_k$  — the angular momentum and  $gE$  — the energy, all at the point  $x_l$ , are the average values of the following dynamical variables

$$(2.1) \quad G(x_l) = \sum_{i=1}^N 2m \delta(R_l^{(i)} - x_l),$$

$$(2.2) \quad I_k(x_l) = \sum_{i=1}^N P_k^{(i)} \delta(R_l^{(i)} - x_l),$$

$$(2.3) \quad M_k(x_l) = \sum_{i=1}^N (\varepsilon_{ktm} R_l^{(i)} P_m^{(i)} + \varepsilon_{ktm} r_l^{(i)} p_m^{(i)}) \delta(R_l^{(i)} - x_l),$$

$$(2.4) \quad E(x_l) = \sum_{i=1}^N \left( \frac{P_k^{(i)} P_k^{(i)}}{4m} + \frac{p_k^{(i)} p_k^{(i)}}{m} + \Phi_{ij} \right) \delta(R_l^{(i)} - x_l) + \frac{1}{2} \sum_{i < j} \Phi_{ij} (\delta(R_l^{(i)} - x_l) + \delta(R_l^{(j)} - x_l)).$$

Performing the computations we obtain:

$$(2.5) \quad g(t, x_l) = \bar{G} = \frac{2m}{\omega} \int F_1(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}) d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)},$$

$$(2.6) \quad g(t, x_l) u_k(t, x_l) = \bar{I}_k = \frac{1}{\omega} \int P_k^{(1)} F_1(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}) d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)},$$

$$(2.7) \quad g(t, x_l) l_k(t, x_l) = \bar{M} = \\ = \frac{1}{\omega} \int \varepsilon_{ktm} (x_l P_m^{(1)} + r_l^{(1)} p_m^{(1)}) F_1(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}) d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)},$$

$$(2.8) \quad g(t, x_l) e(t, x_l) = \bar{E} = \\ = \frac{1}{\omega} \int \left( \frac{P^{(1)2}}{4m} + \frac{p^{(1)2}}{m} + \Phi_{11} \right) F(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}) d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} + \\ + \frac{1}{2\omega^2} \int \Phi_{12} F_2(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}, R_l^{(2)}, r_l^{(2)}, P_l^{(2)}, p_l^{(2)}) d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)}.$$

Moreover, we can introduce the density of the tensor of inertia of the fluid at the point  $x_l$  as the average value of the following dynamical variable

$$(2.9) \quad I_{kn} = \sum_{i=1}^N 2m \varepsilon_{kpq} \frac{1}{2} r_p^{(i)} \varepsilon_{ntq} \frac{1}{2} r_t^{(i)} \delta(R_l^{(i)} - x_l).$$

Thus,

$$(2.10) \quad i_{kn}(t, x_l) = \bar{I}_{kn} = \\ = \frac{m}{2\omega} \int \varepsilon_{kpq} r_p^{(1)} \varepsilon_{ntq} r_t^{(1)} F_1(t, x_l, r_l^{(1)}, P_l^{(1)}, p_l^{(1)}) d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)}.$$

The quantity  $ge$  is defined as

$$(2.11) \quad ge = g \left( \frac{u_k u_k}{2} + \frac{1}{2} m_k \omega_k + \epsilon \right),$$

where  $m_k$  is the intrinsic angular momentum,  $\omega_k$  — the angular velocity of the fluid and  $\epsilon$  — the intrinsic energy.

3. We are now interested in the case of distribution functions which do not differ much from the spatially uniform ones. In the latter case the distribution functions are invariant with respect to the translation of the co-ordinate system:

$$(3.1) \quad F_s(t, R_k^{(1)} + R_k^{(0)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)} + R_k^{(0)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots) = \\ = F_s(t, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots),$$

and give the constant values of hydrodynamical quantities (if these distribution functions do not depend on time). Let us consider the expression

$$(3.2) \quad F_s(t, R_k, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)} - R_k^{(1)} + R_k, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots) = \\ = \Phi_s(t, R_k, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots)$$

and let us notice that

$$(3.3) \quad F_s(t, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots) = \\ = \Phi_s(t, R_k^{(1)}, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots).$$

$\Phi_s$  is invariant with respect to  $R_k^{(1)} + R_k^{(0)}, \dots, R_k^{(s)} + R_k^{(0)}$ . We may conclude that

$$(3.4) \quad F_s(t, R_k^{(1)} + R_k^{(0)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)} + R_k^{(0)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots) = \\ = \Phi_s(t, R_k^{(1)} + R_k^{(0)}, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots),$$

which establishes that the fewer the  $\Phi_s$  changing under the variation  $R_k$ , the fewer the  $F_s$  changing under the translation  $R_k^{(1)} + R_k^{(0)}, R_k^{(2)} + R_k^{(0)}, \dots$

Let us then seek the distribution functions which are near to the spatially uniform distribution functions in the form

$$(3.5) \quad F_s = f_s(t, \mu R_k^{(1)}, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots; \mu), \\ F_1 = f_1(t, \mu R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}; \mu),$$

where  $\mu$  is a small parameter and functions

$$f_s(t, \xi_k, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots; \mu)$$

are asymptotically regular in the vicinity of  $\mu = 0$ . The hydrodynamical quantities (2,5 ÷ 8) can be simply rewritten in the term of  $f_s$ . The equations of motion for the functions  $f_s$  are:

$$(3.6) \quad \frac{\partial f_s}{\partial t} = -\mu \frac{P_k^{(1)}}{2m} \frac{\partial f_s}{\partial \xi_k} + [H_s; f_s] + \\ + \frac{1}{\omega} \int \left[ \sum_{i=1}^s \Phi_{i,s+1}; f_{s+1} \right] d_{(2)} R^{(s+1)} d_{(3)} r^{(s+1)} d_{(3)} P^{(s+1)} d_{(3)} p^{(s+1)}.$$

It must be noted that the Poisson brackets concern the variables  $R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, \dots$  but not the variable  $\xi_k$ .

The properties of symmetry of functions  $f_s$  may be written in the form

$$(3.7) \quad P_{ij} f_s = f_s, \quad (i = 2, 3, \dots, s; j = 2, 3, \dots, s)$$

$$(3.8) \quad P_{1j} f_s(t, \xi_k, R_k^{(1)}, \dots, p_k^{(s)}) = f_s[t, \xi_k - \mu(R_k^{(j)} - R_k^{(1)}), R_k^{(1)}, \dots, p_k^{(s)}] = \\ = f_s - \mu(R_k^{(j)} - R_k^{(1)}) \frac{\partial f_s}{\partial \xi_k} + \frac{\mu^2}{2} (R_k^{(j)} - R_k^{(1)})(R_l^{(j)} - R_l^{(1)}) \frac{\partial^2 f_s}{\partial \xi_k \partial \xi_l} + \mu^3 \dots$$

4. In order to get the general shape of hydrodynamical equations, we differentiate the expressions (2,5 ÷ 8) (rewritten in terms of  $f_s$ ) with respect to time and use the equations of motion (3,6). In this way we obtain successively:

the equation of continuity

$$(4.1) \quad \frac{\partial g}{\partial t} = -\mu \frac{\partial}{\partial \xi_k} g u_k;$$

the equation of momentum

$$(4.2) \quad \frac{\partial g u_l}{\partial t} = -\mu \frac{\partial g u_l u_k}{\partial \xi_k} + \mu \frac{\partial T_{ek}}{\partial \xi_k},$$

where

$$(4.3) \quad T_{lk} = \frac{1}{2m\omega} \int (P_l^{(1)} - 2mu_l)(P_k^{(1)} - 2mu_k) f_1 d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} - \\ - \frac{1}{2\omega^2} \int \frac{\partial \Phi_{12}}{\partial R_l^{(1)}} (R_k^{(2)} - R_k^{(1)}) \left[ f_2 - \frac{\mu}{2} (R_p^{(2)} - R_p^{(1)}) \frac{\partial f_2}{\partial \xi_p} + \mu^2 \dots \right] d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)};$$

the equation of angular momentum

$$(4.4) \quad \frac{\partial g m_l}{\partial t} = -\mu \frac{\partial}{\partial \xi_k} g u_k m_l + \mu \frac{\partial Q_{lk}}{\partial \xi_k} - \varepsilon_{lmn} T_{mn},$$

where

$$(4.5) \quad Q_{lk} = -\frac{1}{2m\omega} \int \varepsilon_{lmn} r_l^{(1)} p_m^{(1)} (P_k^{(1)} - 2mu_k) f_1 d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} - \\ - \frac{1}{\omega^2} \int \varepsilon_{lmn} r_m^{(2)} v' \frac{R_n^{(1)} - R_n^{(2)} + \frac{1}{2}(r_n^{(1)} + r_n^{(2)})}{|R_s^{(1)} - R_s^{(2)} + \frac{1}{2}(r_s^{(1)} + r_s^{(2)})|} \times \\ \times (R_k^{(2)} - R_k^{(1)}) \left[ f_2 + \frac{\mu}{2} (R_p^{(2)} - R_p^{(1)}) \frac{\partial f_2}{\partial \xi_p} - \mu^2 \dots \right] d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)} + \\ + \frac{1}{2\omega^2} \int \varepsilon_{lmn} v' \frac{(r_m^{(1)} + r_m^{(2)})(R_n^{(2)} - R_n^{(1)})}{|R_s^{(1)} - R_s^{(2)} + \frac{1}{2}(r_s^{(1)} + r_s^{(2)})|} \times \\ \times (R_k^{(2)} - R_k^{(1)}) \left[ f_2 - \frac{\mu}{2} (R_p^{(2)} - R_p^{(1)}) \frac{\partial f_2}{\partial \xi_p} + \mu^2 \dots \right] d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)};$$

and the equation of energy

$$(4.6) \quad \frac{\partial g e}{\partial t} = -\mu \frac{\partial}{\partial \xi_k} g u_k e + \mu (u_k T_{kf})_{,f} + \mu (\omega_k Q_{kf} + q_i)_{,i}$$

with

$$(4.7) \quad q_k = -\frac{1}{2m\omega} \int \left\{ \frac{(P_n^{(1)} - 2mu_n)^2}{4m} + \frac{p^{(1)2}}{m} + \Phi_{11} \right\} (P_k^{(1)} - 2mu_k) \times \\ \times f_1^{(0)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} + \\ - \frac{1}{4m\omega^2} \int \Phi_{12} (P_k^{(1)} - 2mu_k) f_2 d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)} - \\ - \frac{1}{8m\omega^2} \int (P_e^{(1)} + P_e^{(2)} - 4mu_e + \\ - 2m\varepsilon_{nte} (R_t^{(2)} - R_t^{(1)}) \omega_n) \frac{\partial \Phi_{12}}{\partial R_e^{(1)}} (R_k^{(2)} - R_k^{(1)}) \left[ f_2 - \frac{\mu}{2} (R_p^{(2)} - R_p^{(1)}) \frac{\partial f_2}{\partial \xi_p} + \right. \\ \left. + \mu^2 \dots \right] d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)} + \\ - \frac{1}{4\omega^2} \int \varepsilon_{nte} (R_t^{(2)} - R_t^{(1)}) \omega_n \frac{\partial \Phi_{12}}{\partial R_e^{(1)}} (R_k^{(2)} - R_k^{(1)}) \left[ f_2 - \right. \\ \left. - \frac{\mu}{2} (R_p^{(2)} - R_p^{(1)}) \frac{\partial f_2}{\partial \xi_p} + \mu^2 \dots \right] d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)}.$$

In these computations we have assumed that the intramolecular and intermolecular potentials, which must be invariant with respect to translations and rotations of co-ordinate systems, have the following forms, respectively:

$$\begin{aligned}
 \Phi_{11} &= \Phi_{11}(|r_k^{(1)}|), \\
 (4.8) \quad \Phi_{12} &= v(|R_k^{(1)} - R_k^{(2)} + \frac{1}{2}(r_k^{(1)} - r_k^{(2)})|) + v(|R_k^{(1)} - R_k^{(2)} + \frac{1}{2}(r_k^{(1)} + r_k^{(2)})|) - \\
 &\quad + v(|R_k^{(1)} - R_k^{(2)} - \frac{1}{2}(r_k^{(1)} + r_k^{(2)})|) + v(|R_k^{(1)} - R_k^{(2)} - \frac{1}{2}(r_k^{(1)} - r_k^{(2)})|).
 \end{aligned}$$

5. Using Bogolyubov's method of derivation of hydrodynamical equations [1], [2], we will seek the functions  $f_s$  in the so-called normal forms, i. e. in such forms in which the functions  $f_s$  depend indirectly on time, but by means of the hydrodynamical quantities  $g, u_k, m_k, \epsilon$ :

$$\begin{aligned}
 (5.1) \quad f_s &= f_s(\xi_k, R_k^{(1)}, r_k^{(1)}, P_k^{(1)}, p_k^{(1)}, R_k^{(2)}, r_k^{(2)}, P_k^{(2)}, p_k^{(2)}, \dots \\
 &\quad \dots; \mu; g(t, \xi_k), u_k(t, \xi_k), m_k(t_1, \xi_k), \epsilon(t, \xi_k)).
 \end{aligned}$$

Thus, the differentiation with respect to time may be written in the form

$$(5.2) \quad \frac{\partial}{\partial t} = \partial_0 + \mu \partial_1 + \mu^2 \partial_2 + \mu^3 \dots,$$

where the operators  $\partial_i$  denote the differentiation with respect to time by means of the hydrodynamical quantities (of specified order with respect to  $\mu$ ).

In connection with this, the equations of motion (3.6) may be split into the following system of equations:

$$\begin{aligned}
 (5.3) \quad [H_s; f_s^{(0)}] + \frac{1}{\omega} \int \left[ \sum_{i=1}^s \Phi_{i,s+1}; f_{s+1}^{(0)} \right] d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} d_{(3)} P^{(s+1)} d_{(3)} p^{(s+1)} = \\
 = \partial_0 f_s^{(0)},
 \end{aligned}$$

$$\begin{aligned}
 (5.4) \quad [H_s; f_s^{(0)}] + \frac{1}{\omega} \int \left[ \sum_{i=1}^s \Phi_{i,s+1}; f_{s+1}^{(0)} \right] d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} d_{(3)} P^{(s+1)} d_{(3)} p^{(s+1)} = \\
 = \partial_1 f_s^{(0)} + \frac{P_k^{(1)}}{2m} \frac{\partial f_s^{(0)}}{\partial \xi_k},
 \end{aligned}$$

where the  $f_s$  is developed in series with respect to  $\mu$ :

$$(5.5) \quad f_s = f_s^{(0)} + \mu f_s^{(1)} + \mu^2 f_s^{(2)} + \dots$$

For the assurance of uniqueness of this development the coefficients must fulfill the following conditions:

$$\begin{aligned}
g(t, \xi_k) &= \frac{2m}{\omega} \int f_1^{(0)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)}, \\
&\quad - \frac{2m}{\omega} \int f_1^{(k)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} = 0 \quad (k = 1, 2, \dots) \\
g(t, \xi_k) u_e(t, \xi_k) &= \frac{1}{\omega} \int P_e^{(1)} f_1^{(0)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)}, \\
&\quad - \frac{1}{\omega} \int P_e^{(1)} f_1^{(k)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} = 0 \quad (k = 1, 2, \dots) \\
g(t, \xi_k) m_l(t, \xi_k) &= \frac{1}{\omega} \int \varepsilon_{ilm} r_l^{(1)} p_m^{(1)} f_1^{(0)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)}, \\
(5.6) \quad &\quad - \frac{1}{\omega} \int \varepsilon_{ilm} r_l^{(1)} p_m^{(1)} f_1^{(k)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} = 0 \quad (k = 1, 2, \dots) \\
g(t, \xi_k) \epsilon(t, \xi_k) &= \frac{1}{\omega} \int \left\{ \frac{(P_k^{(1)} - 2mu_k)^2}{4m} + \frac{\left( p_k^{(1)} - \frac{m}{2} \varepsilon_{ilk} r_l^{(1)} \omega_l \right)^2}{m} + \Phi_{11} \right\} \times \\
&\quad \times f_1^{(0)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} + \frac{1}{2\omega^2} \int \Phi_{12} f_2^{(0)} d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)}, \\
&\quad - \frac{1}{\omega} \int \left\{ \frac{(P_k^{(1)} - 2mu_k)^2}{4m} + \frac{\left( p_k^{(1)} - \frac{m}{2} \varepsilon_{elk} r_l^{(1)} \omega_e \right)^2}{m} + \Phi_{11} \right\} f_1^{(k)} d_{(3)} r^{(1)} d_{(3)} P^{(1)} d_{(3)} p^{(1)} + \\
&\quad + \frac{1}{2\omega^2} \int \Phi_{12} f_2^{(k)} d_{(3)} r^{(1)} \dots d_{(3)} p^{(2)} = 0 \quad (k = 1, 2, \dots).
\end{aligned}$$

In a similar way, we split the symmetry properties of functions  $f_s$  (or the conditions of decrease of correlation). Thus, we obtain:

$$\begin{aligned}
(5.7) \quad P_{1j} f_s^{(0)} &= f_s^{(0)}, \\
P_{1j} f_s^{(1)} &= f_s^{(1)} - (R_k^{(j)} - R_k^{(1)}) \frac{\partial f_s^{(0)}}{\partial \xi_k}, \dots
\end{aligned}$$

6. The solution of the system of Eqs. (5.3) and (5.4) with conditions (5.6) and (5.7) is very difficult. Even in the case of one-atomic fluids the only solutions known are those of the equations of the zero approximation, which correspond to the equilibrium distribution functions [2]. In our case, the equations of the zero approximation (5.3)

differ from the equilibrium equations by the terms  $-\frac{1}{g} \varepsilon_{lmn} T_{mn} \frac{\partial f_s^{(0)}}{\partial m_l}$ .

When the noncentral part of the intermolecular potential is small in comparison with the central part of this potential, the perturbation method may be used in resolving Eqs. (5.3). It is now evident that the solutions of equilibrium equations may be dealt with as the solutions of the zero approximation in respect to the above mentioned perturba-

tion treatment. They will give us the first insight into the problem of coupling of the equations of linear momentum and intrinsic angular momentum.

The solutions of equilibrium equations coming from the Gibbs canonical distribution, we seek these solutions among the following functions:

$$(6.1) \quad f_s^{(0)} = e^{\sum_{i=1}^s \left\{ -\frac{1}{\theta} \left( \frac{P_k^{(i)2}}{4m} + \frac{p^{(i)2}}{m} \right) + p_k^{(i)} (\beta_k + \varepsilon_{mlk} R_l^{(i)} \gamma_m) + \varepsilon_{lkm} r_l^{(i)} p_m^{(i)} \gamma_k \right\}} \varphi_s(R_l^{(1)}, \dots, r_l^{(s)}, \theta, \beta_l, \gamma_l).$$

The functions  $\varphi$  must be solutions of the following equations:

$$\begin{aligned} & \frac{\partial U_s}{\partial r_k^{(i)}} \varphi_s + \theta \frac{\partial \varphi_s}{\partial r_k^{(i)}} + \\ & + \frac{(2\pi\theta m)^3}{\omega} \int \frac{\partial \Phi_{i,s-1}}{\partial r_k^{(i)}} e^{m\theta(\beta_k + \varepsilon_{mlk} R_l^{(s-1)} \gamma_m)^2 - \frac{m\theta}{4} (\varepsilon_{ljk} r_l^{(s-1)} \gamma_j)^2} \varphi_{s-1} d_{(3)} R^{(s-1)} d_{(3)} r^{(s+1)} = 0, \\ (6.2) \quad & \frac{\partial U_s}{\partial R_k^{(i)}} \varphi_s + \theta \frac{\partial \varphi_s}{\partial R_k^{(i)}} + \\ & + \frac{(2\pi\theta m)^3}{\omega} \int \frac{\partial \Phi_{i,s+1}}{\partial R_k^{(i)}} e^{m\theta(\beta_k + \varepsilon_{mlk} R_l^{(s+1)} \gamma_m)^2 + \frac{m\theta}{4} (\varepsilon_{ljk} r_l^{(s+1)} \gamma_j)^2} \varphi_{s+1} d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} = 0, \\ & U_s = \sum_{i \leq j}^s \Phi_{ij}. \end{aligned}$$

The quantities  $\beta_k$ ,  $\gamma_k$  and  $\theta$  may be defined from the conditions (5.7), whence we obtain:

$$\begin{aligned} & u_k = (\beta_k + \varepsilon_{mlk} R_l^{(1)} \gamma_m) \theta \\ & \omega_k = \gamma_k \theta, \\ (6.3) \quad & g \varepsilon = \frac{3\theta}{\omega} (2\pi\theta m)^3 \int e^{m\theta(\beta_k + \varepsilon_{mlk} R_l^{(1)} \gamma_m)^2 + \frac{m\theta}{4} (\varepsilon_{mlk} r_l^{(1)} \gamma_m)^2} \varphi_1 d_{(3)} r^{(1)} + \\ & + \frac{(2\pi\theta m)^3}{\omega} \int \Phi_{11} e^{m\theta(\beta_k + \varepsilon_{mlk} R_l^{(1)} \gamma_m)^2 + \frac{m\theta}{4} (\varepsilon_{mlk} r_l^{(1)} \gamma_m)^2} \varphi_1 d_{(3)} r^{(1)} + \\ & + \frac{1}{2\omega^2} (2\pi\theta m)^6 \int \Phi_{12} e^{\sum_{i=1}^2 \left\{ m\theta(\beta_k + \varepsilon_{mlk} R_l^{(i)} \gamma_m)^2 + \frac{m\theta}{4} (\varepsilon_{mlk} r_l^{(i)} \gamma_m)^2 \right\}} \varphi_2 d_{(3)} r^{(1)} d_{(3)} R^{(2)} d_{(3)} r^{(2)}. \end{aligned}$$

After normalization

$$\begin{aligned} (6.4) \quad & \tilde{\varphi}_s = g^s \left( \frac{\omega}{2m} \right)^s (2\pi\theta m)^{-3s} \Gamma^{-s} e^{-m\theta(\beta_k + \varepsilon_{mlk} R_l^{(1)} \gamma_m)^2} \varphi_s, \\ & \Gamma = \int e^{\frac{m\theta}{4} (\varepsilon_{mlk} r_l^{(1)} \gamma_m)^2} \varphi_1 d_{(3)} r^{(1)}, \end{aligned}$$

the functions  $\nu_s: \tilde{\varphi}_s = \tau_s e^{\nu_s}$  may be introduced ( $\tau_s$  denotes the distribution functions for  $u_k = \omega_k = 0$ ); the equations which they fulfill are:

$$(6.5) \quad \begin{aligned} \tau_s \frac{\partial \nu_s}{\partial r_k^{(i)}} + \frac{1}{2m\theta} \int \frac{\partial \Phi_{i,s+1}}{\partial r_k^{(i)}} [\tilde{g} e^{h_{s+1} - h_s + \nu_{s+1} - \nu_s} - g] \tau_{s+1} d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} &= 0, \\ \tau_s \frac{\partial \nu_s}{\partial R_k^{(i)}} + \frac{1}{2m\theta} \int \frac{\partial \Phi_{i,s+1}}{\partial R_k^{(i)}} [\tilde{g} e^{h_{s+1} - h_s + \nu_{s+1} - \nu_s} - g] \tau_{s+1} d_{(3)} R^{(s+1)} d_{(3)} r^{(s+1)} &= 0, \end{aligned}$$

where

$$\tilde{g} = g e^{-\frac{m}{\theta} u^2} r^{-1}; \quad h_{s+1} - h_s = \frac{m}{\theta} [u_k + \varepsilon_{lk} (R_l^{(s+1)} - R_l^{(1)}) \omega_l]^2 + \frac{m}{4\theta} (\varepsilon_{lk} r_l^{(s+1)} \omega_l)^2.$$

Introducing the functions  $f_s^{(0e)}$  in the form

$$(6.6) \quad f_s^{(0e)} = \left( \frac{\omega}{2m} \right)^s \hat{g}^s (2\pi\theta m)^{-3s} \tau_s e^{i=1} \sum \left\{ -\frac{1}{4m\theta} [P_k^{(i)} - 2mu_k - 2m\varepsilon_{lk} (R_l^{(i)} - R_l^{(1)}) w_l]^2 - \right. \\ \left. - \frac{1}{n\theta} [p_k^{(i)} - \frac{1}{2} m\varepsilon_{lk} r_l^{(i)} w_l]^2 + \frac{m}{\theta} [u_k + \varepsilon_{lk} (R_l^{(i)} - R_l^{(1)}) w_l]^2 + \frac{m}{4\theta} (\varepsilon_{lk} r_l^{(i)} w_l)^2 \right\} + \nu_s$$

the general shape may be calculated of  $T_{lk}^{(0e)}$ ,  $Q_{lk}^{(0e)}$ ,  $q_k^{(0e)}$ , which give the hydrodynamical equations of the zero approximation (in both senses used). We obtain:

$$(6.7) \quad T_{lk}^{(0e)} = -\frac{g}{2m} \theta \delta_{lk} - \frac{1}{2m^2} \tilde{g}^2 \int v' \frac{R_l^{(1)} - R_l^{(2)} + \frac{1}{2}(r_l^{(1)} + r_l^{(2)})}{[R_s^{(1)} - R_s^{(2)} + \frac{1}{2}(r_s^{(1)} + r_s^{(2)})]} \times \\ \times (R_k^{(2)} - R_k^{(1)}) \tau_2 e^{i=1} \sum \left\{ \frac{m}{\theta} [u_k + \varepsilon_{lk} (R_l^{(i)} - R_l^{(1)}) w_l]^2 + \frac{m}{4\theta} (\varepsilon_{lk} r_l^{(i)} w_l)^2 \right\} + \nu_2 d_{(3)} r^{(1)} d_{(3)} R^{(2)} d_{(3)} r^{(2)},$$

$$(6.8) \quad Q_{lk}^{(0e)} = -\frac{\tilde{g}^2}{(2m)^2} \int \varepsilon_{lmn} r_m^{(2)} v' \frac{R_n^{(1)} - R_n^{(2)} + \frac{1}{2}(r_n^{(1)} + r_n^{(2)})}{[R_s^{(1)} - R_s^{(2)} + \frac{1}{2}(r_s^{(1)} + r_s^{(2)})]} \times \\ \times (R_k^{(2)} - R_k^{(1)}) \tau_2 e^{i=1} \sum \left\{ \frac{m}{\theta} [u_k + \varepsilon_{mk} (R_l^{(i)} - R_l^{(1)}) w_m]^2 + \frac{m}{4\theta} (\varepsilon_{mk} r_l^{(i)} w_m)^2 \right\} + \nu_2 d_{(3)} r^{(1)} d_{(3)} R^{(2)} d_{(3)} r^{(2)},$$

$$(6.9) \quad q_k^{(0e)} = -\omega_l Q_{lk}^{(0e)}, \quad \text{where} \quad gm_k = i_{kl} \omega_l.$$

It is worth noting that the expressions  $T_{lk}^{(0e)}$  and  $Q_{lk}^{(0e)}$  are tensors expressed in the terms of  $u_k$  and  $\omega_k$ , which in the case of  $u_k = \omega_k = 0$  give

$$(6.10) \quad T_{lk}^{(0e)} = -p \delta_{lk}; \\ p = \frac{g\theta}{2m} + \frac{\tilde{g}^2}{2m^2} \int v' \frac{R_1^{(1)} - R_1^{(2)} + \frac{1}{2}(r_1^{(1)} + r_1^{(2)})}{[R_s^{(1)} - R_s^{(2)} + \frac{1}{2}(r_s^{(1)} + r_s^{(2)})]} (R_1^{(2)} - R_1^{(1)}) \tau_2 d_{(3)} r^{(1)} d_{(3)} R^{(2)} d_{(3)} r^{(2)},$$

$$(6.11) \quad Q_{ke}^{(0e)} = 0,$$

according to the definition of the pressure. The concrete forms of  $T_{lk}^{(0e)}$  and  $Q_{lk}^{(0e)}$  depend on the concrete forms of  $v$  and  $\nu_s$ .

The quantum mechanical derivation of the hydrodynamical equation will be published in later issues. A detailed discussion of the derivation of hydrodynamical equations will be published elsewhere.

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## Molecular Interaction in the Classical Theory of Light Scattering

by

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The aim of the present paper is to give an account of the effect of interaction between optically anisotropic molecules on the scattering of light in liquids, based on a method previously elaborated [1].

Let us consider a volume  $V$  of the liquid placed at the origin of a system of reference  $(x, y, z)$ , and illuminated by a beam of natural light. We assume the linear dimensions of the volume  $V$  to be small as compared to the incident wavelength, and the volume large enough to contain a great number of molecules.

The component in the direction  $\vec{s}$  of the intensity of the light scattered by the volume  $V$  is given [2] by the equation:

$$(1) \quad I_s = \left(\frac{2\pi}{\lambda}\right)^4 \frac{1}{r^2} \left\langle \left( \Delta \sum_q \vec{p}_q \cdot \vec{s} \right)^2 \right\rangle_{\mathcal{F}},$$

where  $\lambda$  denotes the wave-length *in vacuo* of the incident beam,  $\vec{r}$  — the distance from the origin to the point of observation,  $\vec{p}_q$  — the electric moment of the  $q$ -th molecule induced by the electric vector of the local field  $\vec{E}_0$  of the light wave,  $\vec{s}$  — the unit vector perpendicular to  $\vec{r}$  and determining the position of analyzer, forming the angle  $\Omega$  with the vector  $\vec{E}_0$ .

The magnitude  $\langle (\Delta \sum_q \vec{p}_q \cdot \vec{s})^2 \rangle_{\mathcal{F}}$  represents the statistical mean value of the square of the fluctuation of the  $s$ -component of the electric moment in volume  $V$ . Assuming Gibbs' distribution, the following relation holds for a system in thermodynamical equilibrium:

$$(2) \quad \left\langle \left( \Delta \sum_q \vec{p}_q \cdot \vec{s} \right)^2 \right\rangle_{\mathcal{F}} = C_n \int \int \dots \int \left( \Delta \sum_q \vec{p}_q \cdot \vec{s} \right)^2 e^{-\frac{U_{\mathcal{F}} + U_N}{kT}} d\tau_1 d\tau_2 \dots d\tau_N;$$

here,  $C_n$  denotes the normalization factor,  $k$  — Boltzmann's constant,  $T$  — the Kelvin temperature of the system,  $d\tau_q$  — the configurational

element determining the position and the orientation of the  $q$ -th molecule,  $U_{\mathcal{F}}$  - the potential energy of the system in the external field  $\mathcal{F}$  and  $U_v$  - the potential energy of the interaction existing between the molecules of the liquid.

Let  $N$  be the number of molecules within volume  $V$ ; we then have, for the fluctuation of  $N$  and  $\vec{p}_q$ :

$$(3) \quad \Delta N = N - \langle N \rangle_{\mathcal{F}} \quad \text{and} \quad \Delta \vec{p}_q = \vec{p}_q - \langle \vec{p}_q \rangle_{\mathcal{F}},$$

and Eq. (1) may be rewritten as follows:

$$(4) \quad I_s = \left( \frac{2\pi}{\lambda} \right)^4 \frac{1}{r^2} \left( \langle (\Delta N)^2 \rangle_{\mathcal{F}} \langle \vec{p}_q \cdot \vec{s} \rangle_{\mathcal{F}}^2 + \left\langle \left( \sum_{q=1}^{\bar{N}} \Delta \vec{p}_q \cdot \vec{s} \right)^2 \right\rangle_{\mathcal{F}} \right),$$

where the average number of molecules in  $V$  has been denoted by  $\bar{N}$ .

The magnitude

$$(5) \quad I_s^{(N)} = \left( \frac{2\pi}{\lambda} \right)^4 \frac{1}{r^2} \langle (\Delta N)^2 \rangle_{\mathcal{F}} \langle \vec{p}_q \cdot \vec{s} \rangle_{\mathcal{F}}^2,$$

represents the part of the intensity  $I_s$  scattered as a result of fluctuations of the density of the medium, whereas

$$(6) \quad I_s^{(p)} = \left( \frac{2\pi}{\lambda} \right)^4 \frac{1}{r^2} \left( \left\langle \sum_{q=1}^{\bar{N}} \vec{p}_q \cdot \vec{s} \right\rangle_{\mathcal{F}}^2 - \sum_{q=1}^{\bar{N}} \langle \vec{p}_q \cdot \vec{s} \rangle_{\mathcal{F}}^2 \right),$$

is the intensity of the light scattered as a result of fluctuations of the moment of the molecule, that is, as a result of fluctuations of its anisotropy and orientation and, consequently, of its interaction with other molecules.

The value of the mean square fluctuation of the number of molecules,  $\langle (\Delta N)^2 \rangle_{\mathcal{F}}$ , within the volume  $V$  of the liquid, was computed by M. Smoluchowski [3] and A. Einstein [4], and is given by the equation

$$(7) \quad \langle (\Delta N)^2 \rangle_{\mathcal{F}} = \gamma \bar{N}; \quad \gamma = \beta N_0 kT,$$

wherein  $N_0 = N/V$  is the number of molecules in unit volume, and  $\beta = -\frac{1}{v} \left( \frac{\partial v}{\partial p} \right)_T$  - the coefficient of isothermic compressibility of the medium ( $v$  denotes the specific volume,  $p$  - the pressure). For an ideal gas,  $\gamma = 1$ .

In the weak field of a light wave deformational effects may be neglected. Hence, for the  $q$ -th optically anisotropic molecule we may write

$$(8) \quad \vec{p}_q \cdot \vec{s} = \sum_{ij} a_{ij}^0 \alpha_i^{(q)} \beta_j^{(q)} F_0 = \sum_{ijk} a_{ij}^0 \omega_{ik}^{(pq)} \omega_{jl}^{(pq)} a_k^{(p)} \beta_l^{(p)} F_0,$$

wherein  $a_{ij}^0$  are the components of the tensor of optical polarizability of the molecule,  $a_i^{(q)}, a_k^{(p)}$  and  $\beta_j^{(q)}, \beta_l^{(p)}$  - the directional cosines determining the orientation of the  $q$ -th and  $p$ -th molecule within the system of reference  $(x, y, z)$  with regard to the vectors  $\vec{F}_0$  and  $\vec{s}$ ,  $\omega_{ik}^{(pq)}$  is the cosine of the angle formed by the axis  $i$  of the system  $(X_i^{(p)})$  and the axis  $k$  of  $(X_k^{(q)})$  ( $(X_i^{(p)}), (X_k^{(q)})$  are systems attached to the  $p$ -th and  $q$ -th molecules respectively), and  $i, j, k, l$  - summation indices taking the values 1, 2, 3.

Assuming the tensor of optical polarizability of the molecule  $a_{ij}^0$  to be referred to the principal axes, on isotropic averaging we may rewrite Eqs. (5) and (6), with regard to (7) and (8), as follows:

$$(9) \quad I_s^{(1)} = \left(\frac{2\pi}{\lambda}\right)^4 \frac{\bar{N}}{r^2} \frac{\cos^2 \Omega}{9} \gamma \sum_{ij} a_{ii}^0 a_{jj}^0 F_0^2,$$

$$(10) \quad I_s^{(2)} = \left(\frac{2\pi}{\lambda}\right)^4 \frac{\bar{N}}{r^2} \frac{\cos^2 \Omega + 3}{90} \sum_{ij} a_{ii}^0 a_{jj}^0 \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle F_0^2;$$

the constant index  $p$  refers to the molecule whose position has been fixed.

After substituting (9) and (10) in the fundamental expression (1), a general formula determining the component of the light scattered by the liquid is obtained:

$$(11) \quad I_s = I_0 \left(\frac{2\pi}{\lambda}\right)^4 \frac{\bar{N}}{r^2} \left(\frac{n^2 + 2}{3}\right)^2 \sum_{ij} a_{ii}^0 a_{jj}^0 \times \\ \times \left( \frac{\cos^2 \Omega}{9} \gamma + \frac{\cos^2 \Omega + 3}{90} \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle \right),$$

wherein  $I_0$  denotes the intensity of the incident beam,  $n$  - the optical refractive index and:

$$(12) \quad \langle X(\omega_{ij}) \rangle = C_n' \int \dots \int X(\omega_{ij}) e^{-\frac{U_N}{kT}} d\tau_1 d\tau_2 \dots d\tau_N.$$

It is worth stressing that the expression

$$\left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle$$

appearing in (11) and defining the molecular interaction of the  $N$  molecules present in the volume  $V$  of the liquid also enters the formulae describing the Kerr and Cotton-Mouton constants and those yielding the variation of the permittivity in strong electric and magnetic fields [1], [5]-[7]. Thus, e. g. we have, for the Cotton-Mouton constant,

$$(13) \quad C = \frac{\pi N_0}{15} \left(\frac{n^2 + 2}{3n}\right)^2 \sum_{ij} \left( 3c_{ij,ij}^{0m} - c_{ii,jj}^{0m} + \frac{a_{ii}^0 a_{jj}^0}{kT} \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle \right) \left(\frac{F_m}{H}\right)^2,$$

and for the Kerr constant

$$(14) \quad K = \frac{\pi N_0}{15} \left( \frac{n^2 + 2}{3n} \right)^2 \sum_{ij} \left( 2c_{ijj}^{0e} + \frac{a_{ii}^0 a_{jj}^e}{kT} \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle + \right. \\ \left. + \frac{4b_{ijj}^{0e} \mu_j}{kT} \left\langle \sum_{q=1}^N \omega_{ij}^{(pq)} \right\rangle + \frac{a_{ii}^0 \mu_j^2}{k^2 T^2} \left\langle \sum_{q=1}^N \sum_{r=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pr)} - \omega_{ij}^{(qr)}) \right\rangle \right) \left( \frac{F_e}{E} \right)^2,$$

with  $\mu_i$  denoting the components of the permanent electric moment of the molecule,  $a_{ij}^e$  and  $a_{ij}^m$  — those of the tensor of electric and magnetic polarizability of the molecule, respectively,  $b_{ijj}^{0e}$ ,  $c_{ijj}^{0e}$  and  $c_{ijj,i}^{0m}$  — those of the tensors of electro-optical and magneto-optical deformation of the molecule,  $\vec{E}$  and  $\vec{H}$  the applied electric and magnetic field strength and  $F_e$  and  $F_m$  the local electric and magnetic field.

The degree of depolarization of the light scattered by the liquid (if the primary beam is not polarized) is given by the following general expression obtained from (11):

$$(15) \quad D_n = \frac{6 \sum_{ij} a_{ii}^0 a_{jj}^0 \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle}{\sum_{ij} a_{ii}^0 a_{jj}^0 (10\gamma + 7 \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle)}.$$

Similarly, expression (11) yields a general formula for the light scattering constant of a liquid:

$$(16) \quad R = \frac{1}{2} \left( \frac{2\pi}{\lambda} \right)^4 \left( \frac{n^2 + 2}{9} \right)^2 N_0 \sum_{ij} a_{ii}^0 a_{jj}^0 \left( \gamma + \frac{13}{10} \left\langle \sum_{q=1}^N (3\omega_{ij}^{(pq)} \omega_{ij}^{(pq)} - 1) \right\rangle \right).$$

For molecules having axial symmetry,  $a_{11}^0 = a_{22}^0 \neq a_{33}^0$  and expressions (15) and (16) take the form

$$(17) \quad D_n = \frac{6\delta_0^2 R_{CM}}{5\gamma + 7\delta_0^2 R_{CM}},$$

and

$$(18) \quad R = \frac{\pi^2 (n^2 - 1)^2}{2\lambda^4 N_0} \left( \gamma + \frac{13}{5} \delta_0^2 R_{CM} \right),$$

where

$$(19) \quad R_{CM} = \frac{1}{2} \left\langle \sum_{q=1}^N (3 \cos^2 \theta_{pq} - 1) \right\rangle = 1 + \frac{3}{2} \left\langle \sum_{q=1}^Z (\cos^2 \theta_{pq} - \frac{1}{3}) \right\rangle,$$

represents the so-called correlation factor accounting for the molecular interaction of the molecules of the liquid;  $\theta_{pq}$  denotes the angle between the axis of symmetry of the  $p$ -th and that of the  $q$ -th molecule and  $Z$  — the number of neighbours of a molecule

$$(20) \quad \delta_0 = \frac{a_{33}^0 - a_{11}^0}{a_{33}^0 + 2a_{11}^0}$$

is termed the optical anisotropy of the molecule.

Similarly, from (13) we obtain, for the Cotton-Mouton constant (for  $c_{ij,kl}^{om} = 0$ ):

$$(21) \quad C = \frac{\delta_0 \delta_m \chi_m}{10kTN_0} \cdot \frac{(n^2-1)(n^2+2)}{n^2} \left( \frac{F_m}{H} \right)^2 R_{CM},$$

and from (14) for Kerr's constant (for  $\mu_i = b_{ijk}^{oe} = c_{ijkl}^{oe} = 0$ ):

$$(22) \quad K = \frac{\delta_0 \delta_e \chi_e}{10kTN_0} \cdot \frac{(n^2-1)(n^2+2)}{n^2} \left( \frac{F_e}{E} \right)^2 R_{CM},$$

where

$$(23) \quad \delta_m = \frac{a_{33}^m - a_{11}^m}{a_{33}^m + 2a_{11}^m} \quad \text{and} \quad \delta_e = \frac{a_{33}^e - a_{11}^e}{a_{33}^e + 2a_{11}^e}$$

represent the magnetic and electric anisotropy of the molecule;  $\chi_m$  and  $\chi_e$  are the magnetic and electric susceptibilities and  $R_{CM}$  is defined by Eq. (19).

On neglecting molecular interaction ( $U_N = 0$ ), we have

$$(24) \quad R_{CM} = 1 + \frac{2}{3} Z (\overline{\cos^2 \theta} - \frac{1}{3}) = 1$$

and the formulae (17), (18) and (21), (22) obtained take the form of those already known, applying to gases and liquids [2], [8]-[13].

The correlation factor  $R_{CM}$  given by (19) has been obtained recently in the theory of the effects of molecular orientation [1], [7]. At an earlier date, A. Piekara [9], [10] calculated  $R_{CM}$  in the theory of the Cotton-Mouton effect for dipole pairwise coupling of the molecules in the liquid. Quite recently, A. D. Buckingham [5], in his theory of the second Kerr constant virial coefficient, calculated  $(R_{CM} - 1)$  for various forms of the interaction energy of two molecules.

If the correlation factor  $R_{CM}$  be determined from Eqs. (17), (21) and (22) using experimental values of  $D_n$ ,  $C$  and  $K$ , the magnitude of the effect of molecular interaction on the scattering of light in a liquid is obtained, and thus the consistency of the theory is put to the test.

The following table brings the values of the correlation factor  $R_{CM}$  for a number of polar and non-polar liquids; it will be seen that the values of  $R_{CM}$  as obtained from the various effects are nearly the same.

TABLE

	$C_6H_6$	$C_6H_{12}$	$CS_2$	$C_6H_5CH_3$	$CHCl_3$	$C_6H_5Cl$	$C_6H_5NO_2$
$\mu \cdot 10^{18}$	0	0	0	0.37	1.05	1.6	4.2
$R_{CM}$ from the degree of depolarization of light	0.52	0.45	0.76	0.83	0.91	1.24	1.56
$R_{CM}$ from the Cotton-Mouton constant	0.62	—	0.94	1.02	1.05	1.35	1.68
$R_{CM}$ from Kerr's constant	0.57	0.47	0.83	—	—	—	—

The constant of scattering of light in benzene was given as  $R = 48 \cdot 10^{-1} \text{ cm.}^{-6}$  by C. Carr and B. Zimm [13], whereas measurements by J. Cabannes, recently corroborated by M. Harrand [8], [13] yielded  $R = 28 \cdot 10^{-6} \text{ cm.}^{-1}$  (for  $T = 298^\circ\text{K}$ ,  $\lambda = 4360 \text{ \AA}$ ).

On calculating the coefficient of scattering  $R$  from Eq. (18) we obtained the following values:  $R = 27 \cdot 10^{-6} \text{ cm.}^{-1}$  for  $R_{CM} = 0.52$  as computed from the degree of depolarization of scattered light (17),  $R = 30 \cdot 10^{-6} \text{ cm.}^{-1}$  for  $R_{CM} = 0.62$  as computed from the Cotton-Mouton constant (21), and  $R = 28.5 \cdot 10^{-6} \text{ cm.}^{-1}$  for  $R_{CM} = 0.57$  as computed from Kerr's constant (22). Thus, the present theory will be seen to be in satisfactory agreement with the measurements of J. Cabannes and M. Harrand.

A detailed account of our work will be published in *Acta Physica Polonica*, where a full list of references will be given.

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### Summary

A previously elaborated method [1] was applied for determining the general form of the effect of molecular interaction on the scattering of light in a liquid. This effect is determined by the correlation factor  $R_{CM}$  already obtained in the theory of the effects of molecular orientation [1], [7]. The value of  $R_{CM}$  has been computed for a number of liquids from the degree of depolarization of scattered light  $D_h$ , from the Cotton-Mouton constant  $C$  and from Kerr's constant  $K$ .

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## Application of Pressed Powder Methods to Investigations of Infrared Absorption Spectra of Solids

by

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and I. OLSZEWSKA

*Presented by W. RUBINOWICZ on January 14, 1958*

Existing methods of investigating infrared absorption spectra of many solids are unsatisfactory. For this reason the pressed powder method described by U. Shied and H. Rheinwein [1] and M. M. Stimson and M. J. O'Donnell [2] presents a real progress in infrared absorption spectra research. In order to extend investigation possibilities in this field, the authors applied the pressed powder method \*) to several inorganic and organic solids. A vacuum mould was built for pressing the samples investigated in a Hilger spectrometer H. 668.

The examples of application listed below are only preliminary reports intended to demonstrate further the usefulness of this technique for several different problems.

### Ferroelectric substances

Ferroelectric substances, as for instance barium titanate, were investigated as a powder deposit by Mara et al. [3], as single crystals by C. Hilsum [4], and as single crystals and the new pressed powder technique by J. T. Last [5], who found good agreement between the experimentally determined force constants, compressibilities, and elastic constants, and those found from the Born and von Karman [6] treatment of lattice vibrations.

Using this new technique, the authors compared two kinds of barium titanate samples pressed with KBr, one of which exhibited before milling marked ferroelectric properties (Fig. 1 KBr prism, dashed curve \*\*))

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\*) This paper was presented at the 15th Meeting of the Polish Physical Society held in November 1957 in Wrocław. A detailed account will be published in *Acta Physica Polonica* at a later date.

\*\*) We are indebted for this sample to Professor B. M. Wul of the Institute of Physics, USSR Academy of Sciences, Moscow.

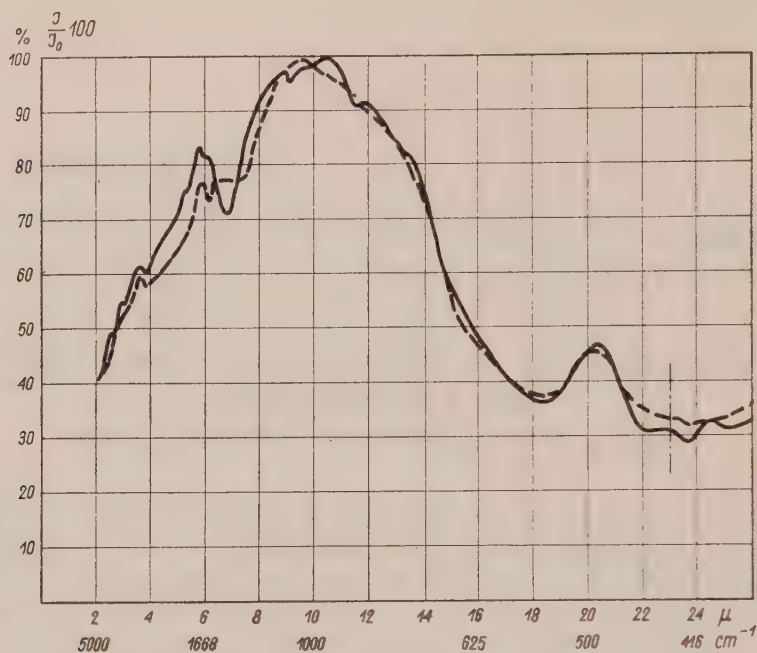


Fig. 1. Comparison between imported  $\text{BaTiO}_3$  (continuous curve) and Wul's sample (---) with KBr prism

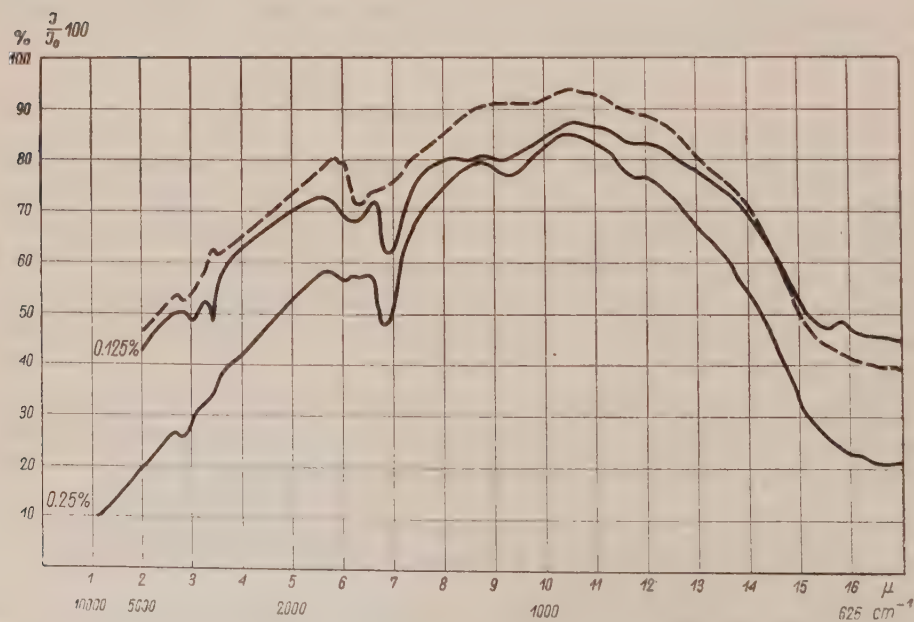


Fig. 2. Comparison between imported  $\text{BaTiO}_3$  (continuous curve) and Wul's sample, with NaCl prism (dashed curve)

as compared with a chemically pure imported  $\text{BaTiO}_3$  in powder form (Fig. 1 continuous curve). Both materials have characteristic "stretching" vibrations at  $545 \text{ cm}^{-1}$  as already found by Last [5]. "Bending" vibration bands could not be exactly determined because no KRS5 prism was available and there was a considerable uncertainty as to the measured transmissions in the neighbourhood of  $400 \text{ cm}^{-1}$  due to low infrared energy in that region. However, the "bending" vibration band could also be determined at approximately  $420 \text{ cm}^{-1}$  (as seen in Fig. 1), which is higher than Last's value of  $400 \text{ cm}^{-1}$ .

Besides these characteristic absorption bands, the imported  $\text{BaTiO}_3$  has also additional bands probably due to impurities, for instance at  $6.95 \mu$ . Such a band was also found by Mara et al. [3], who attributed it to carbonate ion impurities. Fig. 2 shows a comparison of infrared absorption curves investigated with a NaCl prism. The impurity bands in the imported barium titanate are here naturally much more marked than in Fig. 1.

These investigations confirm that the new technique of pressed powders is well suited for determining infrared absorption bands and for finding impurities in ferroelectric materials.

### Polystyrene

In order to determine the usefulness of this technique as applied to plastic materials, the authors investigated polystyrene in pressed powder form as compared to the well-known film technique. Fig. 3 shows

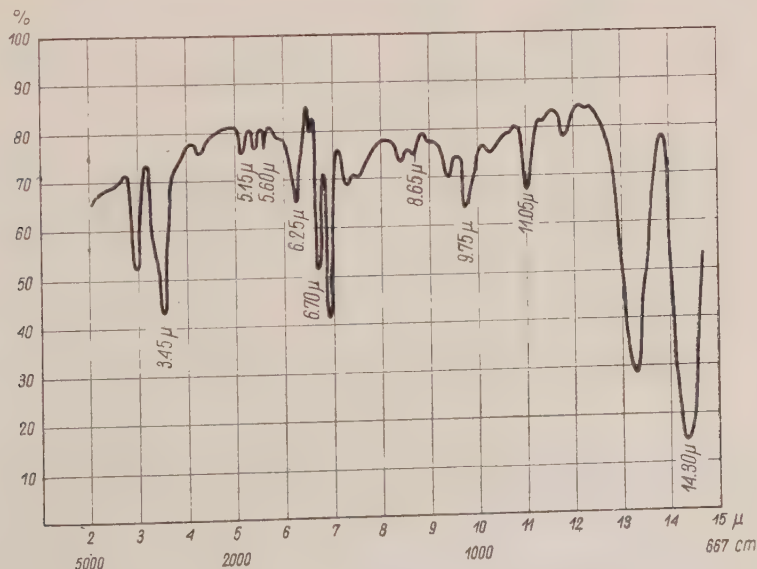


Fig. 3. Polystyrene 0.5% pressed in KBr

the infrared absorption curve for a 0.5% polystyrene powder sample in KBr. All known absorption bands of polystyrene in film form are also present here, although some are perhaps less marked than in films. The authors are of the opinion that this new technique affords possibilities of quantitative comparison of different plastic materials.

### Coals

Infrared absorption spectra investigations were made in England since 1943 by several authors [7]-[13] using different techniques which have not proved very satisfactory. By the end of 1954, R. R. Gordon et al. [14] found in KBr, by the new pressed powder method, infrared absorption spectra for coals with different carbon contents. When approaching this problem the authors had in mind its possible interest to

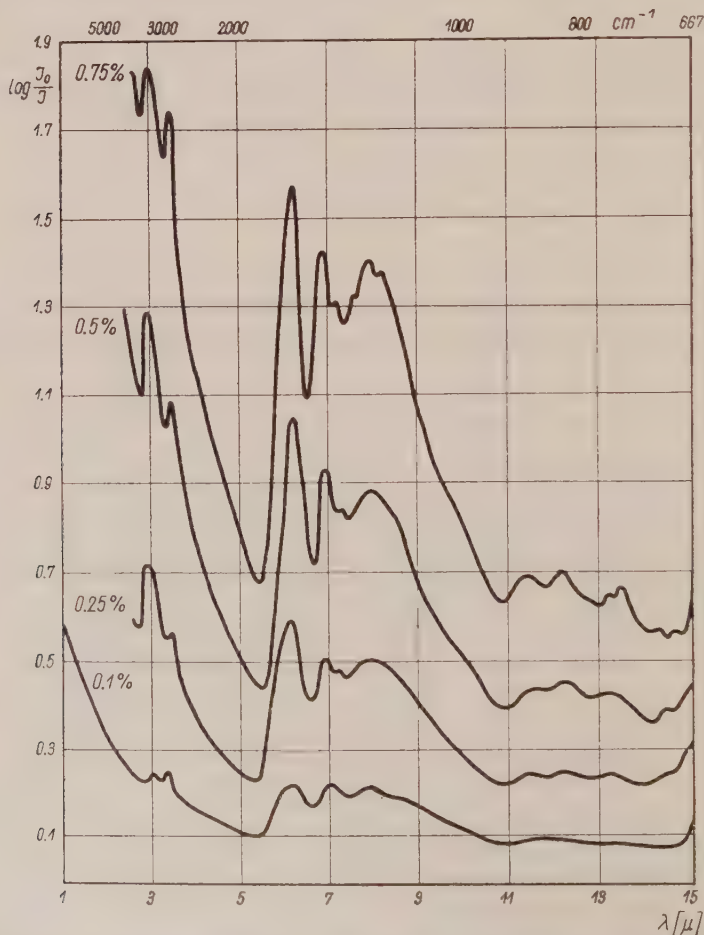


Fig. 4. Infrared absorption spectra of coal from "Rokitnica" coal mine pressed in different percentages with KBr

the Polish coal industry. Fig. 4 shows the infrared absorption spectra of coal taken from the "Rokitnica" coal mine and pressed with potassium bromide at varying concentration. Fig. 5 gives the infrared absorption spectrum of coal from the "General Zawadzki" coal mine.

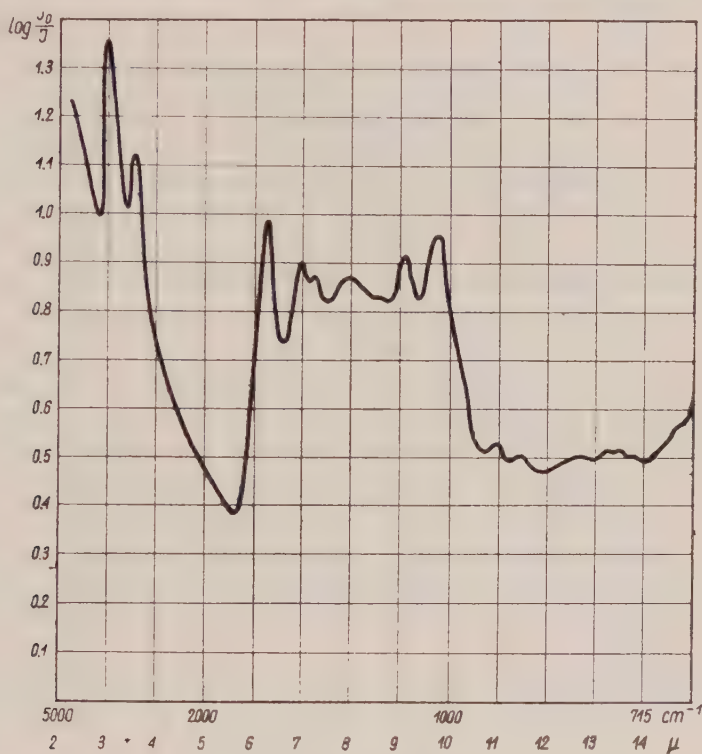


Fig. 5. Infrared absorption spectrum of coal from "General Zawadzki" coal mine pressed with KBr of concentration 0.5%

The authors, together with A. Kujawski, made an attempt at interpreting these absorption bands. Most of the bands mentioned by Cannon [8], [10], [12] and Brown [15] were found in our samples, although the 1100  $\text{cm}^{-1}$  and 1030  $\text{cm}^{-1}$  bands (loc. cit.) were not present in the "Rokitnica" samples.

- ca 3500  $\text{cm}^{-1}$  Hydrogen bonded OH/NH.
- 2900  $\text{cm}^{-1}$  Aliphatic  $\text{CH}_2$  and  $\text{CH}_3$  stretching.
- 1610  $\text{cm}^{-1}$  Aromatic ring frequency.
- ca 1460  $\text{cm}^{-1}$  Deformation frequencies for  $\text{CH}_2$  and  $\text{CH}_3$  groups.
- ca 1380  $\text{cm}^{-1}$  Deformation frequencies for  $\text{CH}_3$  groups.
- 1100  $\text{cm}^{-1}$  C—O stretching of alcohols, linear and cyclic ethers.
- ca 1030  $\text{cm}^{-1}$  C—O stretching of aromatic ethers.
- 750  $\text{cm}^{-1}$  Aromatic C—H out of plane vibration frequencies.

These spectra also show the advantages of the new technique, since the "Nujol" mull method would not permit the detection of bands in the neighbourhood of  $1400\text{ cm}^{-1}$ . It seems that a systematical investigation of different coals could greatly contribute to a deeper knowledge of their characteristics and properties.

#### Biological substances

The application of the new technique to biological substances was studied by Z. Szymanowska \*). The substance studied was inositol, a derivative of cyclohexane. It is a constituent of yeasts, moulds, bacteria and is also found in various plants, animal and human tissues and body fluids. Its physiological role is not yet clarified because of the lack of a specific and direct quantitative method of determination.

It seems that, in bacteria and probably also in certain animal species, inositol plays the part of a vitamin necessary to the growth and development of the organism. The application of the pressed powder technique to this problem could contribute to its solution in developing a method of quantitative analysis for determining inositol.

Fig. 6 shows the infrared absorption spectra curves of inositol in a "Nujol" mull (continuous curve) as compared to a 0.56% sample in potassium bromide (dashed curve). Inositol in a "Nujol" mull was already investigated by J. F. Shay et al. [15] whose results are in good

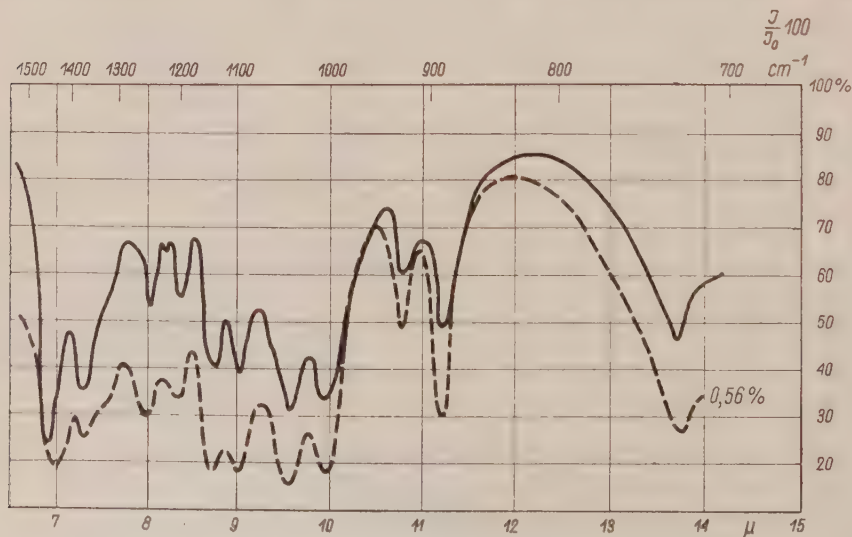


Fig. 6. Comparison of an inositol suspension in "Nujol" (continuous curve) with inositol pressed in KBr (0.56%) broken curve)

\*) Scientific worker of the State Institute of Hygiene, Warsaw, now of the Institute of Biochemistry and Biophysics, Polish Academy of Sciences.

agreement with this investigation (Fig. 6, continuous curve). It can be seen from Fig. 6 that the absorption bands of inositol in "Nujol" coincide well with those of inositol in potassium bromide except for the  $1400\text{ cm.}^{-1}$  region, where they are displaced because of the specific "Nujol" bands.

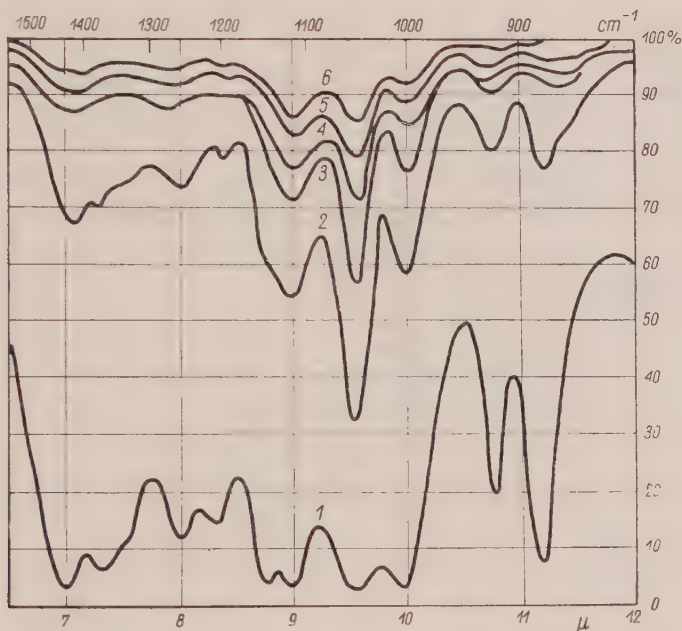


Fig. 7. Infrared absorption spectra of inositol pressed in KBr at different concentrations: (1) 5 mg. 1.0%, (2) 500  $\gamma$  0.1%, (3) 250  $\gamma$  0.05%, (4) 125  $\gamma$  0.025%, (5) 100  $\gamma$  0.02%, (6) 62.5  $\gamma$  0.0125%

Fig. 7 shows absorption spectra of inositol for different concentrations. From these absorption peaks, for  $\lambda = 9.55\ \mu$ , a working curve was derived, and the Lambert and Beer laws could be tested (Fig. 8). In this figure fairly good agreement may be seen with the Lambert and Beer laws for smaller concentrations of inositol.

TABLE I

Test analyses from working curve for  $\lambda = 9.55\ \mu$

No.	Transmission $\frac{I}{I_0}\%$	$\log \frac{I_0}{I}$ experim.	Quantity of inositol		Error %
			from working curve	calculated	
1	42.3	—	332 $\gamma$	333 $\gamma$	0.3
2	65.3	—	155 $\gamma$	166.5 $\gamma$	6.9
3	80.6	—	78 $\gamma$	83.25 $\gamma$	6.3

Test analyses from Lambert-Beer law for  $\lambda = 9.55 \mu$ 

No.	Transmission $\frac{I}{I_0} \%$	$\log \frac{I_0}{I}$	Lambert-Beer law	Calculated	Error %
1	42.3	0.374	350 $\gamma$	333 $\gamma$	0.9
2	65.3	0.185	162 $\gamma$	166.5 $\gamma$	2.7
3	80.6	0.094	82 $\gamma$	83.25 $\gamma$	1.5

Three test analyses were made (Table I). When readings were made from the graph (Fig. 8, straight line) representing Lambert's and

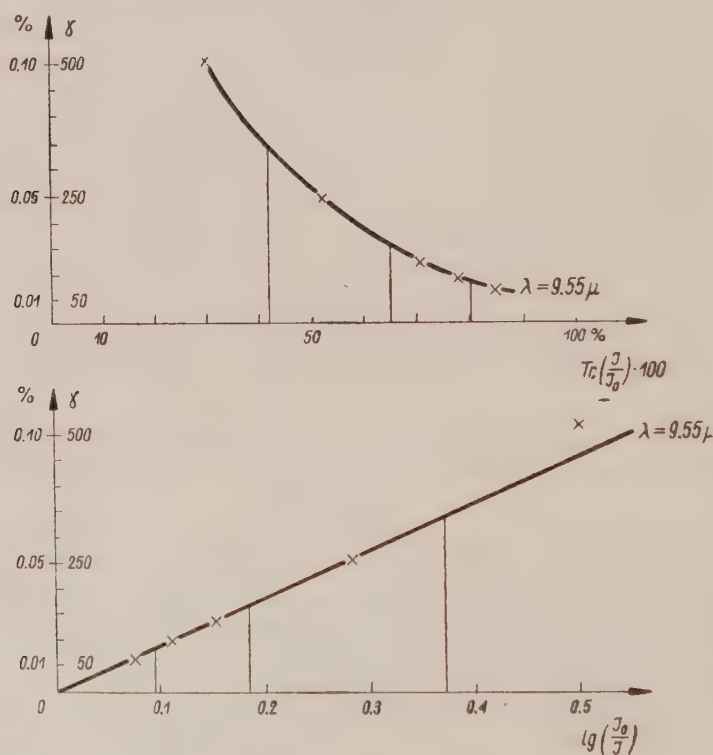


Fig. 8

Beer's law then the maximum error was found to be about 2.7%, as can be seen from Table I. Readings taken from the upper working curve showed greater errors amounting to about 7%. The smallest amount was found to be 30 micrograms of inositol that could be qualitatively detected. It could be quantitatively detected in amounts as low as 60 micrograms. In order to detect quantitatively still smaller amounts of this substance, the volume of the sample should be reduced. The

pressed powder method, as applied to inositol, seems therefore to solve the problem of its quantitative analysis.

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# БЮЛЛЕТЕНЬ

## ПОЛЬСКОЙ АКАДЕМИИ НАУК

### СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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Г. МИЛНЦЕР-ГРУЖЕВСКАЯ, ГРАНИЧНАЯ ТЕОРЕМА О ПРОИЗВОДНОЙ ОБОБЩЕННОГО ИНТЕГРАЛА ПУАССОНА-ВЕЙЕРШТРАССА  
..... стр. 131—133

В настоящей работе автор доказывает граничную теорему:

Пределом пространственной производной обобщенного интеграла Пуассона-Вейерштрасса — когда второй параметр стремится к первому — является производная того-же предела этого интеграла при сохранении предположений, касающихся существования фундаментального решения уравнения с частными производными параболического типа. Кроме того, автор предполагает непрерывность производной плотности вышеупомянутого интеграла и производных коэффициентов квадратической формы этого уравнения в точке, в которой находим предел производной.

А. АЛЕКСЕВИЧ и З. СЕМАДЕНИ, НЕКОТОРОЕ ОБОБЩЕНИЕ ПРОСТРАНСТВ С ДВУМЯ НОРМАМИ. ЛИНЕЙНЫЕ ФУНКЦИОНАЛЫ  
..... стр. 135—139

Дается обобщение понятия пространства с двумя нормами, заменяя более слабую из норм через локально выпуклую топологию. Авторы рассматривают зависимости между линейными функционалами, непрерывными в различном смысле и определяют их общий вид в некоторых случаях. В пространствах с двумя нормами, вообще говоря, неверна теорема о продолжении  $\gamma$ -линейных функционалов; оказывается, однако, что эта теорема верна при условии, что подпространство, из которого продолжаем функционал, является дополнительно линейно-структурным идеалом.

УШО-МО, ЗАМЕТКА О НЕКОТОРЫХ НЕРАВЕНСТВАХ В ТЕОРИИ ФУНКЦИЙ КОМПЛЕКСНОГО ПЕРЕМЕННОГО ..... стр. 141—143

Результаты, полученные в настоящей работе, основываются на применениях некоторых свойств конформного отображения.

В работе рассматриваются следующие вопросы:

1) о некоторых неравенствах, касающихся аналитической функции и ее вещественной части, а также об обобщении неравенства [1], полученного Иво Бабушкой;

- 2) о неравенстве для линейного и двойного интегралов;
- 3) о неравенстве для аналитической функции и ее производной.

А. ПЛИСЬ и Т. ВАЖЕВСКИЙ, УСЛОВИЕ ЕДИНСТВЕННОСТИ РЕШЕНИЙ, ПОЛУЧЕННОЕ ПУТЕМ СРАВНЕНИЯ С ДИФФЕРЕНЦИАЛЬНЫМ УРАВНЕНИЕМ НЕ ОБЛАДАЮЩИМ СВОЙСТВОМ ЕДИНСТВЕННОСТИ РЕШЕНИЙ . . . . . стр. 145—148

В работе дается теорема о единственности решений системы дифференциальных уравнений

$$x'_i = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n).$$

Имеющееся в этом условии сравниваемое уравнение не обязательно должно обладать свойством единственности решений, в противоположность тому, как это имеет место в условиях подобного рода. Свойство единственности решений сравниваемого уравнения заменяется некоторым существенно более общим свойством.

К. МОРЭН, РАЗЛОЖЕНИЕ ПОЛОЖИТЕЛЬНО ОПРЕДЕЛЕННЫХ ЯДЕР ПО ОБОБЩЕННЫМ СОБСТВЕННЫМ ФУНКЦИЯМ. ДИФФЕРЕНЦИРУЕМОСТЬ СПЕКТРАЛЬНОЙ ФУНКЦИИ ГИПОЭЛЛИПТИЧЕСКОГО ОПЕРАТОРА . . . . . стр. 149—155

В работе приводится далеко идущее обобщение теоремы Ю. М. Березанского [1] для положительно определенных ядер (в смысле Л. Шварца). Эта теорема содержит не только результаты, полученные Березанским, но также обобщение существенной теоремы Гординга о дифференцируемости спектральной функции самосопряженного расширения эллиптического оператора.

Доказана также аналогичная теорема для ядер на пространстве двусторонних классов смежности сепарабельной группы Ли, что должно найти применение в общей теории сферических функций.

Я. КРЖИЖ, О ПРОИЗВОДНОЙ ФУНКЦИЙ ОГРАНИЧЕННЫХ И ОДНОЛИСТНЫХ . . . . . стр. 157—159

В связи с еще неопубликованными результатами, полученными В. Яновским при использовании вариационных методов, автор дает простое и элементарное выведение оценки модуля производной для однолистных функций вида  $f(z) = z + a_2 z^2 + \dots$ , ограниченных в единичном круге. Полученная оценка снизу является точной в полном единичном круге, оценка же полученная сверху является точной для  $|z|$ , не превосходящих некоторой величины зависимой от  $M$  ( $M \geq \sup_{|z| < 1} |f(z)|$ ).

Кроме того, автор приводит без доказательства (которое будет дано в Annales UMCS) следующую теорему: производная функции регулярной в круге

$|z| < 1$  и притом такой, что ее значения образуют римановую поверхность с конечной площадью, имеет вид  $o\left(\frac{1}{1-|z|}\right)$  для  $|z| \rightarrow 1$ .

Отсюда легко вытекает, что внутренний радиус односвязной, римановой поверхности, площадь которой конечна, стремится к нулю при приближении к ее границе.

С. МРУВКА, ПРИМЕР НЕ-НОРМАЛЬНОГО ВПОЛНЕ РЕГУЛЯРНОГО ПРОСТРАНСТВА . . . . . стр. 161—163

В работе построено вполне регулярное пространство  $X$ , которое не является нормальным пространством. Пространство  $X$  является небольшим изменением известного пространства Немыцкого [2], но доказательство его полной регулярности — значительно проще. В работе доказано также, что пространство  $X$  является  $Q$ -пространством [4].

А. ШИНЦЕЛЬ, О НЕКОТОРОЙ ПРОБЛЕМЕ КАСАЮЩЕЙСЯ ЧИСЛА ДЕЛИТЕЛЕЙ НАТУРАЛЬНОГО ЧИСЛА . . . . . стр. 165—167

С. Голёмб [1] поставил вопрос — существует ли для каждого натурального числа  $k$  натуральное число  $n$  такое, что

$$\theta(n+1) = \theta(n+2) = \dots = \theta(n+k),$$

где  $\theta(n)$  обозначает число натуральных делителей числа  $n$ .

В работе [2] автор высказал следующую гипотезу N:

N. Если  $f_1(x), f_2(x), \dots, f_s(x)$  — конечная последовательность неприводимых полиномов с целыми коэффициентами, где коэффициент при наивысшей степени переменного всегда положителен, и если нет целого числа  $m > 1$ , делящего при всяком целом  $x$  произведение  $f_1(x) \cdot f_2(x) \cdot \dots \cdot f_s(x)$ , то существует бесконечное множество натуральных чисел  $x$ , для которых все числа  $f_1(x), f_2(x), \dots, f_s(x)$  являются простыми.

Автор доказывает, что из гипотезы N вытекает положительный ответ на вопрос Голёмба.

Ю. МЁДУШЕВСКИЙ, ОБЩЕЕ РЕШЕНИЕ НЕКОТОРОЙ ПРОБЛЕМЫ СИКОРСКОГО . . . . . стр. 169—173

В сообщении доказана следующая теорема: если  $n > 1$  вещественных функций  $f_1, f_2, \dots, f_n$ , определенных на отрезке  $0 \leq x \leq 1$ , непрерывны на нем и ни в каком промежутке не постоянны, то всякая компонента  $S$  множества

$$A_n = \bigcup_{(x_1, x_2, \dots, x_n)} \{f_1(x_1) = f_2(x_2) = \dots = f_n(x_n)\}$$

является регулярной кривой не выше  $2^n$ -ого порядка в строгом смысле, т. е. в смысле существования для любой точки  $p \in S$  строго монотонного се-

мейства открытых областей вида  $G_\varepsilon = \text{Int}(\bar{G}_\varepsilon)$ ; причем  $\varepsilon$  пробегает весь промежуток  $0 < \varepsilon < t$  для некоторого  $t > 0$ , а  $(p) = \prod_{\varepsilon \rightarrow 0} \bar{G}_\varepsilon$  и пересечение  $\text{Fr}(\bar{G}_\varepsilon) \cdot C$  состоит не более, чем из  $2^n$  точек для каждого из этих значений  $\varepsilon$ . Это понятие введено автором настоящего сообщения.

Эта теорема является более общим решением проблемы, поставленной Р. Сикорским [3] для двух функций, чем решение Я. С. Липинского [2]; в этом последнем решении доказана локальная связность компонент множества  $A_2$  и их подконтинуумов.

Оценка  $2^n$  — точная. Если функции  $f_1, f_2, \dots, f_n$ , кроме того, монотонны по отрезкам, составляющим отрезок  $0 \leq x \leq 1$ , то в  $A_n$  имеется лишь конечное число точек ветвления порядка больше 2.

А. РЫБАРСКИЙ, НЕКОТОРЫЙ МЕТОД ЛИНЕАРИЗАЦИИ ДИФ-  
ФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА МАЯТНИКА . . . . . стр. 175—179

Дан метод построения линейного дифференциального уравнения с постоянными коэффициентами, являющегося в некотором смысле наилучшим приближением данного дифференциального уравнения типа маятника. Этот метод основывается на следующей теореме „о сравнении”:

Если функции  $y_1(t), y_2(t)$  выполняют для  $0 \leq t \leq T$  уравнения  $\ddot{y}_1 = f_1(y_1(t))$ ,  $\ddot{y}_2 = f_2(y_2(t))$  и если они равны друг другу в точках  $0 = t_0 < t_1 < \dots < t_{n-1} < T$ , тогда имеется неравенство

$$||\dot{y}_1 - \dot{y}_2|| \leq \frac{w}{w^2 - N} ||f_2(y_1(t)) - f_1(y_1(t))||,$$

где

$$N = - \inf_t \frac{f_2(y_1(t)) - f_2(y_2(t))}{y_1(t) - y_2(t)}, \quad w = \frac{\pi}{\max_k |t_{k-1} - t_k|},$$

причем предполагается, что  $0 \leq N < w^2$ ;  $y_1(t), y_2(t) \in C_2(0, T)$ ;  $f_2(y_1(t)) \in C(0, T)$ , вероятно, за исключением точек  $t_k$ .

Г. ГЕЙЛЬБРОНН, О ПРЕДСТАВЛЕНИИ ГОМОТОПНЫХ КЛАС-  
СОВ РЕГУЛЯРНЫМИ ФУНКЦИЯМИ . . . . . стр. 181—184

В работе доказывается теорема, в которой дается ответ на проблему, поставленную К. Куратовским в работе [1]: Пусть  $A$  — открытое подмножество комплексной плоскости, а  $B$  — множество всех комплексных чисел, различающихся от 0 и от  $\infty$ . В таком случае каждая непрерывная функция, отображающая  $A$  в  $B$  — гомотопна регулярной функции в  $A$ .

Кроме того, в работе даются условия для открытых подмножеств  $A, B$  комплексной плоскости, которые необходимы и достаточны для того, чтобы непрерывная функция, отображающая  $A$  в  $B$  была гомотопна мероморфной функции в  $A$ .

В. ЗОНН и Ю. СТОДУЛКЕВИЧ, ВИДОИЗМЕНЕНИЕ ОСНОВНОГО УРАВНЕНИЯ ЗВЕЗДНОЙ СТАТИСТИКИ, ВЫЗВАННОЕ УЧЕТОМ ПЕРЫВИСТОСТИ МЕЖЗВЕЗДНОЙ МАТЕРИИ . . . . . стр. 185—189

Ввиду того, что количество облаков, пересеченных лучем идущим от звезды (на расстоянии  $r$ ) к наблюдателю следует считать случайной переменной, основное уравнение звездной статистики следует написать в следующем виде:

$$A(m) = \int_0^{\infty} D(r)r^2 dr \int_0^{\infty} \varphi(M)p(x)dx$$

$$M = m - 5 \log r + 5 - \varepsilon x,$$

где  $A(m)$  — число звезд, имеющих видимую величину равную  $m \pm \frac{1}{2} dm$ ,  $M$  — абсолютная величина звезды,  $\varphi(M)$  — функция светимости,  $p(x)$  — вероятность пересечения лучем зрения  $x$  облаков межзвездной материи,  $\varepsilon$  — средняя поглощения в облаке.

Предполагая, что  $p(x)$  является распределением Пуассона и разлагая  $\varphi(M)$  в ряд, получаем приближенное уравнение:

$$A(m) = \int_0^{\infty} D(r)r^2 \varphi(M^0) \left[ 1 + \frac{a_0(r)\varepsilon}{2\varphi(M^0)} \cdot \frac{\partial^2 \varphi(M^0)}{\partial M^2} \right],$$

которым следует пользоваться при определении  $D(r)$  на основании подсчетов звезд  $A(m)$ . Здесь  $a_0(r)$  является средней поглощения на расстоянии  $r$ ;  $M^0 = m - 5 \log r + 5 - \varepsilon x_0$  где  $x_0$  — среднее число пересеченных лучем зрения облаков (являющееся также функцией  $r$ ).

Кроме того, дается метод вычисления распределения поправленной плотности  $D^*(r)$ , на основании распределения плотности  $D(r)$ , вычисленной без учета межзвездного поглощения. Этот метод учитывает дискретный характер межзвездного поглощения при помощи формулы (3).

Соответствующие видоизменения введены также в формулу, на которой опирается известный метод Оорта Вашакидзе определения звездной плотности в больших галактических широтах.

И. ВЕРЛЕ и И. ВЖЕЦИОНКО, ИЗЛУЧЕНИЕ ЧЕРЕНКОВА ПОЛЯРИЗОВАННЫХ ЭЛЕКТРОНОВ . . . . . стр. 191—194

Рассмотрены возможности обнаружения состояния поляризации электрона при помощи измерения поляризации излучения Черенкова. Оказалось, однако, что изменения поляризации излучения Черенкова слишком мало значительны, чтобы их было можно использовать на практике.

М. СУФФЧИНСКИЙ, ТРИЦЕНТРОВЫЕ ИНТЕГРАЛЫ В ЖЕЛЕЗЕ . . . . . стр. 195—198

В настоящей работе рассматриваются трицентровые интегралы, выступающие в матричных элементах энергии вычисленных в приближении ближай-

ших соседей для кубической объемно-центрированной структуры. Для рассматриваемого случая железа, матричные элементы вычислены между состояниями  $s$  и  $d$ . В  $E$ -интегралах, представляющих суммы трицентровых интегралов, принимаются во внимание лишь те члены, в которых сферически-симметричный потенциал центрируется на ионах наиболее близких по отношению к линии, соединяющей два соседних атома. Учитывая преобразования симметрии для кубической объемно-центрированной структуры, выражено шесть  $E$ -интегралов для приближения ближайших соседей при помощи шестнадцати трицентровых интегралов.

Д. ХЛЕБОВСКАЯ, ПО ВОПРОСУ РЕАКЦИИ  $(p, \alpha)$  ДЛЯ ДЕФОРМИРОВАННЫХ ЯДЕР . . . . . стр. 199—201

Выведена формула для дифференциального сечения, для реакции  $(p, \alpha)$  при следующих предположениях: принимается приближение Борна, начальное ядро и конечное ядро сильно деформированы, движение частицы  $\alpha$  в начальном ядре описывается функцией модели независимой частицы.

Р. ЖЕЛЯЗНЫЙ, О ВЫВЕДЕНИИ ГИДРОДИНАМИЧЕСКИХ УРАВНЕНИЙ ДЛЯ КЛАССИЧЕСКИХ СИСТЕМ ДВУХАТОМНЫХ МОЛЕКУЛ . . . . . стр. 203—213

В настоящей работе метод Н. Н. Боголюбова выведения гидродинамических уравнений для классических систем одноатомных молекул применен и обобщен на случай двухатомных молекул. Разработана общая схема выведения гидродинамических уравнений, среди которых имеются здесь уравнения внутреннего момента импульса. Кроме того, при помощи функций распределения, соответствующих каноническому распределению Гиббса, получен общий вид уравнений нулевого приближения, соответствующих уравнениям Эйлера, но с выражениями, которые устанавливают связь между уравнениями импульса и уравнениями внутреннего момента импульса. Вычисление высших приближений, как и в случае одноатомных молекул, оказалось невозможным ввиду таких же трудностей связанных с решением уравнений, определяющих дальнейшие коэффициенты разложения в ряд функций распределения  $fs$ .

С. КЕЛИХ, МЕЖМОЛЕКУЛЯРНОЕ ВЗАИМОДЕЙСТВИЕ В КЛАССИЧЕСКОЙ ТЕОРИИ РАССЕЯНИЯ СВЕТА . . . . . стр. 215—221

В настоящей работе учитывается влияние межмолекулярного взаимодействия на классическое рассеяние света в жидкостях. Автором доказываются формулы для степени деполяризации  $D_n$  рассеянного света и для постоянной рассеяния  $R$ . В этих формулах имеется фактор корреляции  $R_{CM}$ , полученный ранее в теории эффектов Коттон-Мутона и Керра ([1], [7]), и определяющий межмолекулярное взаимодействие в жидкости. Значения фактора корреляции  $R_{CM}$  для некоторых жидкостей, получаемые на основании степени деполяри-

зации  $D_n$  и постоянных Коттона-Мутона и Керра лишь незначительно разнятся друг от друга. Вычисленное автором значение постоянной рассеяния  $R$  для бензола находится в довольно хорошем согласии с результатом измерений Кабанна и Арран.

В. ШИМАНОВСКИЙ, Р. ВИСЬНЕВСКИЙ, Р. ГОЛЭМБЁВ-СКИЙ и И. ОЛЬШЕВСКАЯ, ПРИМЕНЕНИЕ МЕТОДА ПРЕССОВАННЫХ СУСПЕНЗИЙ ДЛЯ ИССЛЕДОВАНИЙ СПЕКТРОВ ПОГЛОЩЕНИЯ В ИНФРАКРАСНОЙ ОБЛАСТИ ДЛЯ ТВЕРДЫХ ТЕЛ . . . . . стр. 223—231

Применяемые до настоящего времени методы исследования спектров поглощения в инфракрасной области являются для многих твердых тел недостаточными. Поэтому, усовершенствованием в этой области является метод прессованных суспензий, разработанный Шидтом и Рейнвейном [1], а также Стимпсоном и О'Донеллем [2]. Авторами настоящего сообщения построена вакуумная аппаратура для прессования проб. При помощи этой аппаратуры был исследован ряд субстанций на спектрометре Гильгера Н.668, а именно: 1) Сопоставлены пробы титаната бария, полученного в советских лабораториях и проявляющего перед его измельчением сильные ферроэлектрические свойства с пробами титаната бария в порошке, полученного в лабораториях других стран. Упомянутые пробы титаната бария спрессованные в бромистом калие проявляли полосы поглощения при  $545\text{ см}^{-1}$  и при ок.  $420\text{ см}^{-1}$ . Подобные результаты были получены И. Т. Ластом [5]. В несоветском титанате бария обнаружены полосы примесей. 2) Полистирол прессованный с бромистым калием проявлял в основном те же полосы поглощения в инфракрасной области, которые были обнаружены в полистироле приготовленном в виде плёнки для исследований. 3) Методом прессованных суспензий исследовался уголь из шахты „Генерал Завадзкий” и сравнивался с углем из шахты „Рокитница”. Уголь из шахты „Генерал Завадзкий” обнаруживал уже известные, на основании исследований другими методами, полосы поглощения в инфракрасной области. В пробах из шахты „Рокитница” полосы при  $1100\text{ см}^{-1}$  и при  $1030\text{ см}^{-1}$  не обнаружены. 4) Этот метод применялся также З. Шимановской для количественного определения инозитола. Полученные полосы инозитола соответствуют известным уже полосам этого вещества, суспендированного в „*Nujol*” [15]. Исследованы спектры поглощения инозитола в КВг для ряда концентраций. На основании этих спектров была определена рабочая кривая и проверен закон Ламберта-Бэра. Три пробных анализа, проведенных при помощи рабочей кривой, дали ок. 7% погрешности, а из закона Ламберта-Бэра получается погрешность ок. 3%. Наименьшее количество инозитола, которое могло быть обнаружено количественно, составляло ок.  $60\text{ мкг}$ . Качественно можно было обнаружить ок.  $30\text{ мкг}$ .



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